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Research Article

Complexity of Some Types of Cyclic Snake Graphs

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Abstract

The number of spanning trees in graphs (networks) is a crucial invariant, and it is also an important measure of the reliability of a network. Spanning trees are special subgraphs of a graph that have several important properties. First, *T* must span *G*, which means it must contain every vertex in graph *G*, if *T* is a spanning tree of graph *G*. *T* needs to be a subgraph of *G*, second. Stated differently, any edge present in *T* needs to be present in *G* as well. Third, *G* is the same as *T* if each edge in *T* is likewise present in *G*. In path-finding algorithms like Dijkstra's shortest path algorithm and A* search algorithm, spanning trees play an essential part. In those approaches, spanning trees are computed as component components. Protocols for network routing also take advantage of it. In numerous techniques and applications, minimum spanning trees are highly beneficial. Computer networks, electrical grids, and water networks all frequently use them. They are also utilized in significant algorithms like the min-cut max-flow algorithm and in graph issues like the travelling salesperson problem. In this paper, we use matrix analysis and linear algebra techniques to obtain simple formulas for the number of spanning trees of certain kinds of cyclic snake graphs.

Keywords

Cyclic Snakes, Subdivision, Edge Contraction, Spanning Trees

1. Introduction

In this paper, we give some basic definitions. We deal with simple and finite undirected graphs G = (V, E), where V is the vertex set and E is the edge set. For a graph G, a spanning tree in G is a tree that has the same vertex set as G. The number of spanning trees in G, also called the complexity of the graph, denoted by τ (G), is a well-studied quantity (for a long time) and appears in a number of applications. The most notable application fields are network reliability [1–3], enumerating certain chemical isomers [4], and counting the number of Eulerian circuits in a graph [5]. A classical result of Kirchhoff [6] can be used to determine the number of spanning trees for G = (V, E). Let $V=\{v_1, v_2, ..., v_n\}$, then the Kirchhoff matrix H defined as n ×n characteristic matrix H = D −A, where D is the diagonal matrix of the degrees of G and A is the adjacency matrix of G, H = [aij] defined as follows: (i) aij = −1, vi and vj are adjacent and i ≠ j, (ii) aij equals the degree of vertex vi if i = j, and (iii) aij = 0 otherwise. All of the co-factors of H are equal to τ(G). There are other methods for calculating τ(G). Let $\mu_1 \ge \mu_2 \ge .... \ge \mu_p$ denote the eigenvalues of H matrix of p point graph. Then it is easily shown that μ_p = 0. Furthermore, Kelmans and Chelnokov [7] have shown that, $\tau(G) = 1/p\prod_{k=1}^{p-1}\mu_k$. The formula for the number of span-

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ning trees in a d-regular graph G can be expressed as $\tau(G) = 1/p\prod_{k=1}^{p-1}(d-\mu_k)$ where $\lambda_0 = d, \lambda_1, \lambda_2, ..., \lambda_{p-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph. However, for a few special families of graphs, there exist simple formulas that make it much easier to calculate and determine the number of corresponding spanning trees, especially when these numbers are very large.

One of the first such results is due to Cayley [8], who showed that for a complete graph with n vertices, K_n has n^{n-2} spanning trees that he showed $\tau(K_n) = n^{n-2}, n \geq 2$. $\tau(K_{p,q}) = p^{q-1}q^{p-1}$ $p,q \geq 1$ where $K_{p,q}$ is the complete bipartite graph with bipartite sets containing p and q vertices, respectively. It is well known, as in e.g., [9, 10]. Another result is due to Sedlacek [11], who derived a formula for the wheel on n + 1 vertices, W_{n+1}, which is formed from a cycle Cn on n vertices by adding a vertex adjacent to every vertex of C_n. In particular, he showed that $\tau(W_{n+1}) = (3+\sqrt{5}/2)^n + (3-\sqrt{5}/2)^n - 2$, for $n \geq 3$. Sedlacek [12] also derived a formula for the number of spanning trees in a Mobius ladder. The Mobius ladder M_n is formed from cycle C_{2n} on 2n vertices labelled $v_1, v_2, ..., v_{2n}$ by adding an edge $v_i v_{i+n}$ for every vertex v_i where $i \leq n$.

The number of spanning trees in M_n is given by $\tau(M_n) = n/2[(2+\sqrt{3})^n + (2-\sqrt{3})^n + 2)]$ for $n \geq 2$. Another class of graphs for which an explicit formula has been derived is based on a prism [13, 14]. Let the vertices of two disjoint length cycles be labelled $v_1, v_1, \ldots, v_n$ in one cycle and $w_1, w_1, \ldots, w_n$ in the other. The prism R_n is defined as the graph obtained by adding to these two cycles all the edges of the form $v_i w_{i.}$. The number of spanning trees in R_n is given by the following formula $n/2[(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2]$. For more results, it is suggested to see these articles [15-28].

Chio's condensation is a method for evaluating an $n{\times}n$ determinant in terms of (n − 1) × (n − 1) determinants; see [29]:

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{Vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{Vmatrix}$$

2. Dodgson's Condensation Method

Dodgson's condensation method computes determinants of size $n \times n$ by expressing them in terms of those of size (n − 1) × (n − 1), and then expresses the latter in terms of determinants of size (n−2)×(n−2), and so on (see [30]).

This method is based on Dodgson and Chio's method, but the difference between them is that this new method is resolved by calculating 4 unique determinants of (n − 1) × (n − 1) order, (which can be derived from determinants of $n \times n$ order, if we remove first row and first column or first row and last column or last row and first column or last row and last column, elements that belong to only one of the unique determinants), and one determinant of (n − 2) × (n − 2) order which is formed from $n \times n$ order determinant with elements $a_{i,j}$ with i, j = 1, n, on condition that the determinant of (n − 2) × (n − 2) = 0.

$$A = \frac{1}{\begin{pmatrix} a_{22} & \cdots\cdots & a_{2,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,2} & \cdots\cdots & a_{n-1,n-1} \end{pmatrix}} \begin{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} \end{pmatrix} & \begin{pmatrix} a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n} \end{pmatrix} \\ \vdots & \vdots \\ \begin{pmatrix} a_{21} & \cdots & a_{2,n-1} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n-1} \end{pmatrix} & \begin{pmatrix} a_{22} & \cdots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{n,2} & \cdots & a_{nn} \end{pmatrix} \end{pmatrix}$$

3. Complexity of Some Types of Cyclic Snake Graphs

Lemma 4.1: Given a simple graph G with n vertices, its laplacian matrix L_{nxn} is defined as: $L = D - A$ where D is the degree matrix and A is the adjacency matrix of the graph. In the case of directed graphs, either the in degree or the out degree might be used, depending on the application. The elements of L are given by

$$L_{i,j} := \begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

where deg(v_i) is degree of the vertex i

Definition 4.1 [31]:

A triangular snake (or Δ_k-*snake*) is a connected graph in which all blocks are triangles and the block-cut-point graph is a path.

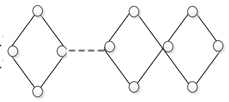
Definition 4.2: The $kc_n - snake$ is called linear, if the block-cut-vertex graph of $kc_n - snake$ has the property that the distance between any two consecutive cut-vertices is $[\frac{n}{2}]$ Definition 4.3: The graphs (m, k) C_4 as the family of graphs kC_4-snake where every block has m copies of C_4 with two non- adjacent vertices in common, where the number of blocks is denoted by k.

Definition 4.4: The subdivision of a graph G is obtained by subdividing every edge of G exactly once.

The Main Results:

Theorem 5.1 The number of spanning trees of the linear

$kc_4 - snake$

$\tau(\ kc_4 - snake\ ) = \tau\ ($  $)$

$\tau(\ kc_4 - snake\ ) = 4^k, k \geq 2,$ where k is the number of blocks.

Proof. By induction, prove at m = 1, $\tau(G)$ is a 2* 2 matrix with both rows the same:

$$\tau(G) = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = 4.$$

We try to prove at m=k that is

Straightforward induction using properties of determinants and Dodgson and Chio method and applying Lemma 2.1, we have:

$$\tau(G) = \begin{bmatrix}
4 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 & -1 & -1 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & -1 & -1 & -1 & -1 & 0 & \vdots \\
\vdots & \ddots & 4 & -1 & \ddots & -1 & -1 & -1 & 0 & 0 & \vdots & \vdots \\
\vdots & \ddots & -1 & 2 & -1 & \ddots & 0 & 0 & 0 & \vdots & \vdots & \vdots \\
\vdots & \ddots & 0 & -1 & \ddots & -1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & -1 & 0 & -1 & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & -1 & -1 & 0 & \cdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots \\
-1 & -1 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \ddots & 0 \\
-1 & 0 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & -1 \\
0 & \vdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 2
\end{bmatrix}_{2k*2k} = \frac{1}{2*4^{k-1}}\begin{vmatrix} 3*4^{k-1} & -4^{k-1} \\ -4^{k-1} & 3*4^{k-1} \end{vmatrix} = \frac{(3*4^{k-1})^2 - (-4^{k-1})^2}{2*4^{k-1}} = \frac{8*4^{2k-2}}{2*4^{k-1}} = 4^k\ k \geq 2, Where\ \text{k}\ is\ the\ number\ of\ blocks\ .$$

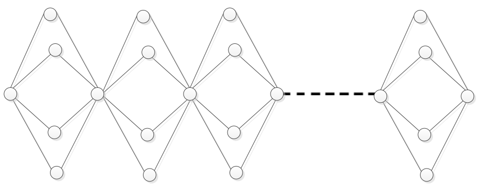
$$S = \begin{vmatrix}
4 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 & -1 & -1 & -1 & 0 \\
0 & \ddots & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \iddots & \iddots & \iddots & \iddots & \vdots \\
\vdots & \ddots & 4 & -1 & \ddots & -1 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \vdots \\
\vdots & \ddots & -1 & 2 & -1 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \ddots & \ddots & -1 & \ddots & -1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \cdots & -1 & \ddots & -1 & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \iddots & -1 & 0 & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \iddots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
-1 & \iddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
-1 & \iddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
-1 & \iddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
-1 & \iddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 2
\end{vmatrix}_{(2k-2)*(2k-2)} = 2*4^{k-1}$$

$$A = \begin{vmatrix}
4 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 & -1 & -1 \\
0 & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & -1 & -1 & -1 & 0 \\
\vdots & \ddots & 4 & -1 & 0 & -1 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & -1 & 2 & -1 & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & -1 & \ddots & -1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & \ddots & -1 & \ddots & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & \vdots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & \vdots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & 0 & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & -1 & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & -1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
-1 & -1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
-1 & -1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\
-1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 2
\end{vmatrix}_{(2k-2)*(2k-2)} = 3*4^{k-1},$$

$$B = B^T = \begin{vmatrix}
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 & -1 & -1 & 0 \\
4 & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 & -1 & -1 & -1 & 0 & \vdots \\
0 & \ddots & \ddots & \cdots & \cdots & 0 & -1 & -1 & -1 & -1 & 0 & \cdots & \vdots & \vdots \\
\vdots & \ddots & 4 & -1 & 0 & -1 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & -1 & 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & 0 & -1 & \ddots & -1 & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & -1 & 0 & -1 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & -1 & \vdots & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & -1 & -1 & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
-1 & -1 & 0 & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
-1 & -1 & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
-1 & -1 & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
-1 & 0 & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 2 & -1
\end{vmatrix}_{(2k-2)*(2k-2)} = 4^{k-1}$$

$$C = \begin{vmatrix}
4 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 & -1 & -1 & -1 & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & -1 & -1 & -1 & -1 & 0 & \cdots & \cdots & \vdots \\
\vdots & \ddots & 4 & -1 & 0 & -1 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \ddots & -1 & 2 & -1 & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \ddots & 0 & -1 & \ddots & -1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & -1 & \ddots & -1 & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & -1 & -1 & \vdots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & -1 & -1 & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
-1 & -1 & 0 & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
-1 & -1 & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
-1 & 0 & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
-1 & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\
0 & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & -1 & 2
\end{vmatrix}_{(2k-2)*(2k-2)} = 3*4^{k-1}$$

Theorem 5.2 The number of spanning trees of the (2, *k*) *C*4-snake

τ((2, k) C4-snake) = τ()

$\tau((2, \mathrm{k})\ \mathrm{C4\text{-}snake}) = (32)^k\ k \geq 3$

Proof. By induction, prove at $m = 1$, $\tau(G)$ is a 2* 2 matrix with both rows the same:

$$\tau(G) = \frac{1}{24}\begin{bmatrix} 32 & 16 \\ 16 & 32 \end{bmatrix} = 32.$$

We try to prove at *m*=*k* that is

Straightforward induction using properties of determinants and Dodgson and Chio method and applying Lemma 2.1, we have:

$$\tau(G)=\begin{bmatrix}
2 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 2 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & 0 \\
-1 & -1 & 4 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
-1 & -1 & 0 & 8 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & -1 & -1 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & -1 & 0 & 2 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & 0 & -1 & \vdots & \ddots & 2 & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & -1 & \vdots & \vdots & 0 & 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & 0 & \vdots & \vdots & -1 & -1 & \ddots & -1 & \cdots & \cdots & -1 & 0 & -1 & -1 & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & -1 & \vdots & \vdots & 0 & 0 & -1 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & -1 & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 & -1 & 0 & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & -1 & \vdots & \vdots & 0 & \ddots & -1 & 0 & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & -1 & -1 & 8 & -1 & -1 & -1 & -1 & 0 & -1 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & \ddots & 0 & \ddots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & \ddots & 2 & 0 & -1 & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & 0 & 0 & 2 & -1 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & -1 & -1 & 4 & -1 \\
0 & 0 & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 2
\end{bmatrix}_{5k*5k}$$

$$=\frac{1}{\frac{3}{2}(32)^k}\begin{vmatrix} (32)^k & \frac{1}{2}(32)^k \\ \frac{1}{2}(32)^k & \frac{7}{4}(32)^k \end{vmatrix}=(32)^k\ k\geq 3, \textit{Where k is the number of blocks}.$$

$$-\tau(G)=\frac{1}{S}\begin{vmatrix} A & B \\ B^T & C \end{vmatrix}$$

$$=\frac{1}{\begin{vmatrix}
2 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & 0 \\
-1 & 4 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
-1 & 0 & 8 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
0 & -1 & -1 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & -1 & -1 & 0 & 2 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & 0 & -1 & \vdots & \ddots & 2 & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & \vdots & \vdots & 0 & 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & 0 & \vdots & \vdots & -1 & -1 & \ddots & -1 & \cdots & \cdots & -1 & 0 & -1 & -1 & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & \vdots & \vdots & 0 & 0 & -1 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 & -1 & 0 & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & -1 & \vdots & \vdots & 0 & \ddots & -1 & 0 & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & -1 & -1 & 8 & -1 & -1 & -1 & -1 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & \ddots & 0 & \ddots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & \ddots & 2 & 0 & -1 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & 0 & 0 & 2 & -1 \\
0 & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & -1 & -1 & 4
\end{vmatrix}}
\begin{vmatrix}
\begin{vmatrix}
2 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 2 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
-1 & -1 & 4 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
-1 & -1 & 0 & 8 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & -1 & -1 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & -1 & 0 & 2 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & 0 & -1 & \vdots & \ddots & 2 & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & -1 & \vdots & \vdots & 0 & 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & 0 & \vdots & \vdots & -1 & -1 & \ddots & -1 & \cdots & \cdots & -1 & 0 & -1 & -1 & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & -1 & \vdots & \vdots & 0 & 0 & -1 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & -1 & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 & -1 & 0 & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & -1 & \vdots & \vdots & 0 & \ddots & -1 & 0 & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & -1 & -1 & 8 & -1 & -1 & -1 & -1 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & \ddots & 0 & \ddots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & \ddots & 2 & 0 & -1 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & 0 & 0 & 2 & -1 \\
0 & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & -1 & -1 & 4
\end{vmatrix}
&
\begin{vmatrix}
0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
2 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
-1 & 4 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
-1 & 0 & 8 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
0 & -1 & -1 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & -1 & -1 & 0 & 2 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & 0 & -1 & \vdots & \ddots & 2 & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & \vdots & \vdots & 0 & 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & 0 & \vdots & \vdots & -1 & -1 & \ddots & -1 & \cdots & \cdots & -1 & 0 & -1 & -1 & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & \vdots & \vdots & 0 & 0 & -1 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 & -1 & 0 & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & -1 & \vdots & \vdots & 0 & \ddots & -1 & 0 & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & -1 & -1 & 8 & -1 & -1 & -1 & -1 & 0 & -1 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & \ddots & 0 & \ddots & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & \ddots & 2 & 0 & -1 & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & 0 & 0 & 2 & -1 & 0 \\
0 & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & -1 & -1 & 4 & -1
\end{vmatrix}
\\
\begin{vmatrix}
0 & 2 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & 0 \\
-1 & -1 & 4 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
-1 & -1 & 0 & 8 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & -1 & -1 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & -1 & 0 & 2 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & 0 & -1 & \vdots & \ddots & 2 & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & -1 & \vdots & \vdots & 0 & 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & 0 & \vdots & \vdots & -1 & -1 & \ddots & -1 & \cdots & \cdots & -1 & 0 & -1 & -1 & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & -1 & \vdots & \vdots & 0 & 0 & -1 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & -1 & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 & -1 & 0 & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & -1 & \vdots & \vdots & 0 & \ddots & -1 & 0 & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & -1 & -1 & 8 & -1 & -1 & -1 & -1 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & \ddots & 0 & \ddots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & \ddots & 2 & 0 & -1 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & 0 & 0 & 2 & -1 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & -1 & -1 & 4 \\
0 & \cdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1
\end{vmatrix}
&
\begin{vmatrix}
2 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & 0 \\
-1 & 4 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
-1 & 0 & 8 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
0 & -1 & -1 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & -1 & -1 & 0 & 2 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & 0 & -1 & \vdots & \ddots & 2 & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & \vdots & \vdots & 0 & 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & 0 & \vdots & \vdots & -1 & -1 & \ddots & -1 & \cdots & \cdots & -1 & 0 & -1 & -1 & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & \vdots & \vdots & 0 & 0 & -1 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 & -1 & 0 & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & -1 & \vdots & \vdots & 0 & \ddots & -1 & 0 & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & -1 & -1 & 8 & -1 & -1 & -1 & -1 & 0 & -1 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & \ddots & 0 & \ddots & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & \ddots & 2 & 0 & -1 & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & 0 & 0 & 2 & -1 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & -1 & -1 & 4 & -1 \\
0 & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 2
\end{vmatrix}
\end{vmatrix}$$

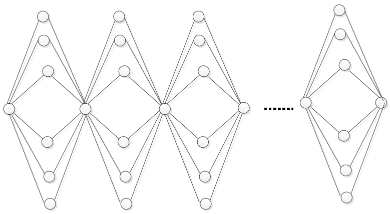
$$\frac{1}{s}=\frac{1}{\begin{vmatrix}
2 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\
-1 & 4 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\
-1 & 0 & 8 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\
0 & -1 & -1 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\
\vdots & -1 & -1 & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\
\vdots & 0 & -1 & 0 & \ddots & 2 & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\
\vdots & \vdots & -1 & \vdots & 0 & \ddots & 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\
\vdots & \vdots & -1 & \vdots & \vdots & -1 & -1 & \ddots & \ddots & -1 & -1 & -1 & 0 & -1 & -1 & 0 & \cdots & \vdots\\
\vdots & \vdots & 0 & \vdots & \vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\
\vdots & \vdots & -1 & \vdots & \vdots & \vdots & \vdots & -1 & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & -1 & \vdots & \vdots & \vdots & \vdots & -1 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & -1 & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \vdots & \vdots & -1 & \ddots & 8 & -1 & -1 & -1 & -1 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & -1 & \vdots & \vdots & \cdots & 0 & -1 & 2 & 0 & \cdots & 0 & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & -1 & \vdots & \vdots & \vdots & \vdots & -1 & 0 & 2 & \ddots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & -1 & \vdots & \ddots & 2 & 0 & -1\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & -1 & 0 & \cdots & 0 & 2 & -1\\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & -1 & -1 & 4
\end{vmatrix}},$$

$$A=\begin{vmatrix}
2 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\
0 & 2 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\
-1 & -1 & 4 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\
-1 & -1 & 0 & 8 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\
0 & 0 & -1 & -1 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\
\vdots & \vdots & -1 & -1 & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\
\vdots & \vdots & 0 & -1 & \vdots & 0 & 2 & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\
\vdots & \vdots & \vdots & -1 & \vdots & \vdots & 0 & 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\
\vdots & \vdots & \vdots & 0 & \vdots & \vdots & -1 & -1 & \ddots & \ddots & -1 & -1 & -1 & 0 & -1 & -1 & 0 & \cdots & \vdots\\
\vdots & \vdots & \vdots & -1 & \vdots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\
\vdots & \vdots & \vdots & -1 & \vdots & \vdots & \vdots & \vdots & -1 & \ddots & \ddots & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\
\vdots & \vdots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & -1 & 0 & \ddots & \ddots & \ddots & -1 & 0 & \cdots & \cdots & \cdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & -1 & \vdots & 0 & \ddots & \ddots & \ddots & 0 & \cdots & \cdots & \cdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & -1 & -1 & 8 & -1 & -1 & -1 & -1 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & -1 & \vdots & \cdots & 0 & \cdots & -1 & 2 & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & -1 & \vdots & \vdots & \vdots & \vdots & -1 & 0 & 2 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & -1 & 0 & 0 & 2 & 0 & -1\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & -1 & 0 & 0 & 0 & 2 & -1\\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & -1 & -1 & 4
\end{vmatrix}$$

$$
B=\begin{vmatrix}
0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
2 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
-1 & 4 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
-1 & 0 & 8 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
0 & -1 & -1 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & -1 & -1 & 0 & 2 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & 0 & -1 & \vdots & \ddots & 2 & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & \vdots & \vdots & 0 & 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & 0 & \vdots & \vdots & -1 & -1 & \ddots & -1 & \cdots & \cdots & -1 & 0 & -1 & -1 & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & \vdots & \vdots & 0 & 0 & -1 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 & -1 & 0 & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & -1 & \vdots & \vdots & 0 & \ddots & -1 & 0 & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & -1 & -1 & 8 & -1 & -1 & -1 & -1 & 0 & -1 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & \ddots & 0 & \ddots & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & \ddots & 2 & 0 & -1 & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & 0 & 0 & 2 & -1 & 0 \\
0 & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & -1 & -1 & 4 & -1
\end{vmatrix},\;
B^T=\begin{vmatrix}
0 & 2 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & 0 \\
-1 & -1 & 4 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
-1 & -1 & 0 & 8 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & -1 & -1 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & -1 & 0 & 2 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & 0 & -1 & \vdots & \ddots & 2 & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & -1 & \vdots & \vdots & 0 & 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & 0 & \vdots & \vdots & -1 & -1 & \ddots & -1 & \cdots & \cdots & -1 & 0 & -1 & -1 & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & -1 & \vdots & \vdots & 0 & 0 & -1 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & -1 & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 & -1 & 0 & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & -1 & \vdots & \vdots & 0 & \ddots & -1 & 0 & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & -1 & -1 & 8 & -1 & -1 & -1 & -1 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & \ddots & 0 & \ddots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & \ddots & 2 & 0 & -1 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & 0 & 0 & 2 & -1 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & -1 & -1 & 4 \\
0 & \cdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1
\end{vmatrix},
$$

$$
C=\begin{vmatrix}
2 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & 0 \\
-1 & 4 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
-1 & 0 & 8 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
0 & -1 & -1 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & -1 & -1 & 0 & 2 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & 0 & -1 & \vdots & \ddots & 2 & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & \vdots & \vdots & 0 & 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & 0 & \vdots & \vdots & -1 & -1 & \ddots & -1 & \cdots & \cdots & -1 & 0 & -1 & -1 & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & \vdots & \vdots & 0 & 0 & -1 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 & -1 & 0 & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & -1 & \vdots & \vdots & 0 & \ddots & -1 & 0 & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & -1 & -1 & 8 & -1 & -1 & -1 & -1 & 0 & -1 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & \ddots & 0 & \ddots & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & \ddots & 2 & 0 & -1 & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & 0 & 0 & 2 & -1 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & -1 & -1 & 4 & -1 \\
0 & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & 0 & 0 & 0 & -1 & 2
\end{vmatrix}
$$

Theorem 5.3 The number of spanning trees of the (3, *k*) *C*4-snake

$\tau((3,\text{ k) C4-snake}) = \tau($  $)$

$$\tau((3,\text{ k) C4-snake}) = (192)^k \; k \geq 2$$

Proof.

$$\tau(G) = \begin{bmatrix}
2 & 0 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 2 & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 2 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-1 & -1 & -1 & 6 & 0 & -1 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-1 & -1 & -1 & 0 & 12 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & -1 & -1 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & -1 & -1 & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & -1 & -1 & \vdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & 0 & -1 & \vdots & \vdots & 0 & 2 & 0 & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & -1 & \vdots & \vdots & \vdots & 0 & 2 & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & -1 & \vdots & \vdots & \vdots & 0 & 0 & 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & 0 & \vdots & \vdots & \vdots & -1 & -1 & -1 & \ddots & -1 & \cdots & \cdots & \cdots & \cdots & -1 & 0 & -1 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & -1 & \vdots & \vdots & \vdots & 0 & 0 & 0 & -1 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots\;\vdots & \vdots & \vdots & \vdots & -1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \cdots & \cdots & \cdots & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\
\vdots & \vdots & \vdots & \vdots & -1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \ddots & \cdots & \cdots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \cdots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & -1 & -1 & -1 & 12 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & -1 & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & -1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & -1 & 0 & 2 & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & -1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & -1 & \vdots & \ddots & 2 & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & -1 & \vdots & \ddots & \ddots & 2 & \ddots & \cdots & -1 & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & -1 & 0 & \ddots & \ddots & \ddots & 2 & 0 & -1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & -1 & -1 & 6 & -1 & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\
0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2
\end{bmatrix}_{7k*7k}$$

Straightforward induction using properties of determinants and Dodgson and Chio method and applying Lemma 2.1, we have:

$$\tau(G) = \frac{1}{\frac{11}{12}*(192)^k} \begin{vmatrix} (192)^k & \frac{1}{2}(192)^k \\ \frac{1}{2}(192)^k & \frac{7}{6}(192)^k \end{vmatrix}$$

$$\tau(G) = \frac{\frac{7}{6}(192)^{2k} - \frac{1}{4}(192)^{2k}}{\frac{11}{12}*(192)^k} = \frac{\frac{11}{12}*(192)^{2k}}{\frac{11}{12}*(192)^m} = (192)^k \; k \; \geq 2, \textit{where} \text{ k is the number of blocks.}$$

4. Conclusion

The number of spanning trees in graphs (networks) is an important invariant, and it is also an important measure of reliability of a network. In this paper we derived simple formulas of the complexity, number of spanning trees, of some types of cyclic snake graphs using linear algebra and matrix analysis techniques.

Conflicts of Interest

The authors declare no conflicts of interes.

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