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Existence and Numerical Investigation of Monkey-Pox Mathematical Model by Natural Adomain Decomposition Method

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Abstract: In this paper, studied the mathematical model concerning the transmission of Monkey-Pox disease. A class viral disease that mostly occurs in west and central Africa, transmitted from animals into human is belonging to the Small-pox family known is Monkey-pox infections disease. According to the scientist the primary best of the proposed disease is still in doubt. The proposed model will be investigate for the purpose of both qualitative and numerical solutions. At the early stage of this study, investigate the existence of proposed model. In this connection, the authors developed the desired condition of existence and stability for consider model by using the tools of analysis. At the second phase of this research work,the author investigated the numerical solutions for the consider Monkey-pox transmission diseases model. For numerical investigation, the authors use the tool of well know semi-analytical techniques known as Natural Transform coupled with Adomain Decomposition Method. The consider techniques are powerful tools for of obtaining approximate solutions of differential equation or system of differential equations. The proposed techniques base on recursive scheme for solutions of system of differential equations. For the authenticity and accuracy of obtain solutions, the obtain solutions are visualized graphically to desired the dynamical behavior of desired results with the help of Mathematica. That show the proposed method is best tools for solution of differential equations.

Keywords: Monkey-pox, Natural Transform, Adomain Decomposition Method

1. Introduction

Differential equations (DEs) are mathematical tools used for modeling of varieties of real word problems. The concern tools are very help in the modeling of transmission diseases models now days [16]. Besides the diseases models, DEs are variety of applications, including in the field of physics are motion, electrodynamics, in engineering are control system and electrical circuits, in biology are population dynamics, pharmacokinetics, in economics are macroeconomics, games theory, in chemistry are chemical kinetics and reaction engineering, in computer science in machine learning in biological science and finance are portfolio optimization. DEs offer a strong method for evaluation and projecting dynamics behavior of system in a variety of fields. In theory of DEs, two ideas that are extremely important are existence and uniqueness of the solutions [19].

Researcher are keen interest to investigate different aspect of DEs, due it variety of applications in different fields of sciences. The significant aspects of these studies are existence, uniqueness and stability analysis of DEs, system of DEs, coupled, system of DEs, hybrid DEs and it coupled system, disease model and many more [26, 27]. In some circumstances, it is quite difficult to obtained exact analytical solution of DEs, in such a situation the authors focusing on numerical investigation of those problems. The authors are well explore different features of DEs, system of DEs, coupled and hybrid DEs. The investigation of system for biological models are quite new and are rarely investigate by the authors for existence and numerical solution [20]. In this connection, the consider area of research is quite new and needs further attention of research to investigate of numerical solution of disease models. Nearly almost dynamical and biological problems are non-linear in natures and occurs in form of system or coupled system of DEs. An important tool of mathematical investigated is mathematical modeling (MM). The area devoted to MM is relatively new one in the modern sciences, engineering areas and it has attracted a significant amount of interest from modern scientists [4]. MM is a method that is utilized in a variety of professional areas for the purpose of predicting, describing, and analyzing situations that occur in the real world through the application of mathematical concepts [3]. In point of fact, MM is a process that involves giving a mathematical representation of a number of different problems that need to be solved in the fields of biological research, environmental science, physical science, engineering, economics, social science, natural systems and getting insight a problem [5, 37, 38]. The researcher have the opportunity to make decisions and gain a more in-depth understanding of complicated systems through the use of mathematical simulations and analysis. The modeling of infectious diseases is one of the most important aspects of MM [21], it offers a variety of benefits, including the ability to predict and predicted the outbreak of infectious diseases to help policymakers in formulating potential policies, to determine the peak of the disease, to evaluate the impact of interaction and to estimate the number of future cases. More help can be provided regarding the manner in disease will spread and the people who will be most at risk within the community [7].

Throughout the duration history, infectious diseases have continuously presented major risks to human health, and also for proper mathematics of human society. This is an international issue and effected on a variety of different regions all over the world. Experts have put in a lot of effort to develop a variety of mathematical tools that may be used to identify the complicated patterns of transmission and produce more effective management plans. In-fact that these diseases are unexpected and complex [1]. Within the past few years, there has been an important portion of progress made in the field of MM associated with infectious diseases. Due to this improvement, fresh possibilities have been opened up for the purpose of understanding as well as controlling the variety of complicated issues that are brought about by these diseases [8]. Through the help of these brilliant mathematical tools, the author are able to gain a better understanding of how diseases are transmitted, who is most susceptible to them, and how to maximize treatments such as vaccinations and quarantine restrictions in order to lessen the impact of these diseases. They are also helpful in the prediction of cases the efficient management of resources and eventually the saving of lives. In basically, these new mathematical techniques work as essential instruments in the ongoing fight against infectious diseases. They enable us to create strategies that are more precise and effective in order to protect the health of the general public and preserve the social unity of different communities [10].

A class viral disease that mostly occurs in west and central Africa transmitted from animals into human is belonging to the Small-pox family is known is Monkeypox (MP) infections disease. According to the scientist the primary best of the proposed disease is still in doubt, but the virus that causes MP is a member of the orthopox virus family and is transmitted by monkeys [28]. Animals, particularly rats such as squirrels and monkeys, are the primary vectors by this disease is transmitted. Human-to-human transmission can occur through a variety of methods including breathing in small amounts of water through the respiratory system [29, 33, 34], direct contact with the skin or feces of an infected person and having control of possibly infected items, such as clothing and furniture. A high temperature, aching muscles, back pain, swollen glands, chills, and a feeling of stress are some of the symptoms that it might cause [9]. Within five to twenty-one days of infection, the symptoms of this virus begin to appear. This disease is characterized by the appearance of a visible breakouts it is typically begins on the face and spreads to other parts of the body. In addition to pimples, infections and expanding the rash can progress through a number of the stages [11]. The death rate connected to it is between one percent to ten percent, and those with weakened immune systems are particularly at danger [13]. The countries of Nigeria, Cameroon, Central Africa, and West Africa are the ones that are most usually affected by the proposed disease. One of the most important factors in the spread of this disease is the presence of tourists who have traveled from an area that is infected [30]. Considering the fact that there is no medical treatment for MP the smallpox vaccine does offer some protection against the disease. In order to effectively manage infections it is necessary to implement public health follows such as controlling contact and removing those who are afflicted [12]. Maintaining a clean environment and reducing close contact with animals that may be affected are two more ways to reduce the likelihood of contracting an infection [31, 32]. Analyzed the MP disease model that was presented below.

The novelty of this paper is focusing on the MP model under Ordinary Differential Equations (ODEs), that will be solved through Natural Adomain Decomposition method (NADM). Furthermore, introduced the death parameter in existence MP model i.e. α_i, where $i = 1, 2, 3$ in human compartment this model. The previous model ignore the effect of death rate related to the infection, that is unrealistic. Therefore, in this study the researcher introduced the disease related death rate, modeling the dynamics of underlying disease. With this, it is expected that the model revolution the hidden insides of MP disease model.

We will use the tools of analysis to establish condition for existence, and numerical investigation of proposed model by well known approximate techniques as known as NADM. The proposed MP model is given by

$$\begin{aligned}
\frac{dS_h}{dt} &= \Pi_h - \Phi(t)S_h - \eta_h S_h, \\
\frac{dE_h}{dt} &= \Phi(t)S_h - (\tau_1 + \tau_2 + \eta_h + \alpha_1)E_h, \\
\frac{dI_h}{dt} &= \tau_1 E_h - (\psi + \eta_h + \rho_1 + \alpha_2)I_h, \\
\frac{dQ_h}{dt} &= \tau_2 E_h - (\phi + \rho_2 + \eta_h + \alpha_3)Q_h, \\
\frac{dR_h}{dt} &= \psi I_h + \phi Q_h - \eta_h R_h, \\
\frac{dS_r}{dt} &= \Pi_r - \Theta(t)S_r - \eta_r S_r, \\
\frac{dE_r}{dt} &= \Theta(t)S_r - (\eta_r + \xi_r)E_r, \\
\frac{dI_r}{dt} &= \xi_r E_r - \eta_r I_r.
\end{aligned} \tag{1}$$

Table 1. *Description of the Compartments.*

$S_h(\mathrm{t})$	Susceptible Community of Humans
$E_h(\mathrm{t})$	The community of Exposed Humans
$I_h(\mathrm{t})$	Community of Individually Affected Humans
$Q_h(\mathrm{t})$	Individual Quarantined Humans Population
$R_h(\mathrm{t})$	Community of Recovered Humans
$S_r(\mathrm{t})$	Susceptible Rodents Community (Monkeypox)
$E_r(\mathrm{t})$	Exposed Rodents Community (Monkeypox)
$I_r(\mathrm{t})$	Infected Rodents Community (Monkeypox)

While the initial conditions are
$S_h(0)\ 0,\ E_h(0) \geq 0,\ I_h(0) \geq 0,\ \ Q_h(0) \geq 0,\ R_h(0) \geq 0,$ $S_r(0) \geq 0, \quad E_r(0) \geq 0, \quad I_r(0) \geq 0.$
where $\Phi(t) = \theta_1 I_r + \theta_2 I_h$ and $\Theta(t) = \theta_r I_r$.

Table 2. *Description of the Compartments.*

$S_h(\mathrm{t})$	Susceptible Community of Humans
$E_h(\mathrm{t})$	The community of Exposed Humans
$I_h(\mathrm{t})$	Community of Individually Affected Humans
$Q_h(\mathrm{t})$	Individual Quarantined Humans Population
$R_h(\mathrm{t})$	Community of Recovered Humans
$S_r(\mathrm{t})$	Susceptible Rodents Community (Monkeypox)
$E_r(\mathrm{t})$	Exposed Rodents Community (Monkeypox)
$I_r(\mathrm{t})$	Infected Rodents Community (Monkeypox)

This paper has been divided into six sections, the first section has been focused on deep introduction about MP model. The second section of this paper provides basic definition, lemmas and theorems related to the problem. The third section has been devoted to existence, uniqueness and stability analysis for the MP model (1). In the fourth section of this paper, the tools of NADM, to obtain the desired numerical solution for proposed model. In the fifth section, provided a graphical representation of the obtained results to describe dynamics of the proposed disease model (1). While in the last section, a conclusion and future work have been added to related work.

Table 3. *Description of parameters.*

π_h	Rate of healthy community recruitment.
η_h	Underlying mortality rate.
θ_1	The rate of effective transmission of the virus from a virus-infected rodent to a human.
θ_2	Relationships that are effective in spreading the monkeypox virus throughout humans.
τ_1	An individual from the affected class.
τ_2	Everybody in the exposed class is placed under quarantine.
ψ, ϕ	Both the infected and quarantined classes recover after infection at different rates.
ρ_1	The virus claims the lives of members of the affected classes at a rate.
ρ_2	The inmates have a higher disease-related death rate.
α_i	Death Rate ($\alpha_1, \alpha_2, \alpha_3$).
Π_r	The rate of births determines the increase in the population of animals (rodents).
η_r	Reductions brought on by the normal rate of death.
θ_r	Rodents' frequency of contact with infected and uninfected.
ξ_r	Impact of infection occurs in exposed animals.
$\mathfrak{N}^+$	Natural Transform

2. Preliminaries

2.1. Theorems

Theorem 2.1. Let B be a convex subset of $\mathcal{Z}$, assume that operator equation satisfies the following condition for each $\mathcal{F}\mathrm{J}(u) \in \mathrm{B}$ and for each $u \in \mathrm{B}$:

1. $\mathcal{F}\mathrm{J}(u) \in B$, and for each $u \in B$
2. $\mathcal{F}\mathrm{J}(\mathrm{t})$ is contraction,
3. $\mathcal{F}\mathrm{J}(\mathrm{t})$ is compact,

Thus, $\mathcal{F}\mathrm{J}(\mathrm{t})$ has at least one solution.

Theorem 2.2. (*Relatively Compact*) A subset A of a metric space W is said to be relatively compact , if its closure $\overline{A}$ is compact in W.

Theorem 2.3. (*Norm*) Let $\wp$ be a vector space over a field $\mathbb{G}$. A function $\|\cdot\| : \wp \to \mathbb{G}$ is called a norm on $\wp$, if it satisfies the following properties for all vectors $w, \hbar \in \wp$ and scalars $\lambda \in \mathbb{G}$:

1. Non-negativity: $\|w\| \geq 0$ and $\|w\| = 0$, if and only if $w = \mathbf{0}$ (the zero vector).
2. Scalar Multiplication: $\|\lambda w\| = |\lambda|\|w\|$.
3. Triangle Inequality: $\|w + \hbar\| \leq \|w\| + \|\hbar\|$.

Theorem 2.4. (*Completely Continuous*) A linear operator $T : W \to Y$ between normed spaces W and Y is said to be completely continuous, if it takes bounded sets in W to sets that are relatively compact in Y.

Theorem 2.5. (*Arzel?Ascoli*) Let "W" be a compact metric space, and let $\mathcal{F}$ be a family of real-valued continuous functions on W. If $\mathcal{F}$ is uniformly bounded and equi-continuous, then there exists a sub-sequence of $\mathcal{F}$ that converges uniformly on W to a continuous limit function.

2.2. Lemma

Lemma 2.1. (*Lipschitz condition*) A function $\Im(t, \hbar)$ satisfies a Lipschitz condition in the variable $\hbar$ on a set $M \subset \mathbb{G}^2$, if a constant $H > 0$ exists such that

$$|\Im(t, \hbar_1) - \Im(t, \hbar_2)| \leq H|\hbar_1 - \hbar_2|,$$

whenever $(t, \hbar_1), (t, \hbar_2) \in M$ and H is called the Lipschitz constant.

2.3. Definitions

Definition 2.1. (*Gamma function*) The famous gamma function is define by

$$\Gamma(m) = \int_0^\infty \varrho^{-\varpi} \varpi^{m-1} d\varpi, \;\; m \in \Re^+.$$

for Re$(m) > 0$, where Re(m) denotes the real part of Z. The gamma function satisfies the recurrence relation $(\Gamma(m+1) = m\Gamma(m))$.

Definition 2.2. (*Natural Transform*) If $f(\mathbf{t})$ be a piecewise continuous function and $(\mathrm{t}) \geq 0$, then the NT of $\mathrm{f}(\mathbf{t})$ is defined as

$$\mathfrak{N}^+[\mathrm{f}(\mathbf{t})] = \mathcal{F}(w, s) = \int_0^\infty e^{-st}\mathrm{f}(wt)\mathrm{d}t; \qquad s > 0, \; w > 0, \tag{2}$$

where "s" and "w'' are transform variables.

Equation (2) is the combination of Laplace and sumudo transform. For $w = 1$, equation (2) converges to Laplace transform. Now for s=1, equation (2) converges to Sumudo transform.

Property 2.6. *(The Natural transform of the derivative of a function).* The NT of the derivative of a function $f(w)$ is given by:

$$\mathfrak{N}^+[f'(w)] = \frac{s}{u}\mathcal{F}(w, s) - \frac{f(0)}{u}.$$

Where $\mathfrak{N}^+[f'(w)]$ denotes the Natural Transform of $f'(w)$, $\mathcal{F}(w, s)$ is a function depending on "w" and "s".

Property 2.7. *(Linearity of Natural transform)* Let $f(t), g(t)$ be the piece-wise continuous functions and α, β are section numbers

$$\mathfrak{N}^+\left[\alpha f(\mathbf{t}) + \xi g(\mathbf{t})\right] = \alpha\mathfrak{N}^+\left[f(\mathbf{t})\right] + \xi\mathfrak{N}^+\left[g(\mathbf{t})\right].$$

3. Adomian Decomposition Method

Adomian decomposition method combine with Natural transform give an influential method known as NADM. In numerous situation,the authors paid more attention to NADM, due to it accuracy and reliability. Therefore, a strong motivation has been given to the aforesaid techniques and it was used as a tool for numerical solution consider of MP model. For the computation analysis of system [21], the non-linear terms $\mathrm{I}_r(\mathbf{t})\mathrm{S}_h(\mathbf{t})$, $\mathrm{I}_h(\mathbf{t})\mathrm{S}_h(\mathbf{t})$ and $\mathrm{I}_r(\mathbf{t})\mathrm{S}_r(\mathbf{t})$ are decomposed by Adomian decomposition method as.

$$\mathrm{I}_r(\mathbf{t})\mathrm{S}_h(\mathbf{t}) = \sum_{i=0}^{\infty} \mathrm{A}_i(\mathbf{t}),$$

$$\mathrm{I}_h(\mathbf{t})\mathrm{S}_h(\mathbf{t}) = \sum_{i=0}^{\infty} \mathrm{B}_i(\mathbf{t}),$$

$$\mathrm{I}_r(\mathbf{t})\mathrm{S}_r(\mathbf{t}) = \sum_{i=0}^{\infty} \mathrm{C}_i(\mathbf{t}).$$

Where each $\mathrm{A}_i(t)$, $\mathrm{B}_i(t)$ and $\mathrm{C}_i(t)$ represents the Adomian polynomials and is defined as;

$$
\begin{aligned}
A_i(t) &= \frac{1}{\Gamma(i+1)} \frac{\mathbf{d^i}}{d\boldsymbol{\lambda^i}} \left[\sum_{n=0}^{i} \boldsymbol{\lambda^n} I_{rn}(t) \sum_{n=0}^{i} \boldsymbol{\lambda^n} S_{hn}(t) \right]_{\lambda=0}, \\
B_i(t) &= \frac{1}{\Gamma(i+1)} \frac{\mathbf{d^i}}{d\boldsymbol{\lambda^i}} \left[\sum_{n=0}^{i} \boldsymbol{\lambda^n} I_{hn}(t) \sum_{n=0}^{i} \boldsymbol{\lambda^n} S_{hn}(t) \right]_{\lambda=0}, \\
C_i(t) &= \frac{1}{\Gamma(i+1)} \frac{\mathbf{d^i}}{d\boldsymbol{\lambda^i}} \left[\sum_{n=0}^{i} \boldsymbol{\lambda^n} I_{rn}(t) \sum_{n=0}^{i} \boldsymbol{\lambda^n} S_{rn}(t) \right]_{\lambda=0}.
\end{aligned} \tag{3}
$$

4. Existence and Stability Proposed Model

This section, of the study focuses on the existence of a solution for the proposed MP Model (1). To meet the required conditions for this solution, the utilize tools of analysis.

4.1. Existence Theory

In this section, the necessary conditions were establish for the proposed model. Will use the tools of analysis to investigate the formation MP mathematical model (1). We can express the consider model (1) in the form of ODEs as.

$$
\begin{cases} \dfrac{d}{dt}\mathrm{J(t)} = \mathrm{B}(t, \mathrm{J(t)}) \quad t \in [0, T], \\ \mathrm{J}(0) = \mathrm{J}_0. \end{cases} \tag{4}
$$

Where $\mathrm{J(t)}$, $\mathrm{J_0(t)}$ and $\mathrm{B}(t, \mathrm{J(t)})$ as define as

$$
\mathrm{J(t)} = \begin{cases} S_h(\mathrm{t}) \\ \mathbb{E}_h(\mathrm{t}) \\ \mathbb{I}_h(\mathrm{t}) \\ \mathbb{Q}_h(\mathrm{t}) \\ \mathbb{R}_h(\mathrm{t}) \\ S_r(\mathrm{t}) \\ \mathbb{E}_r(\mathrm{t}) \\ \mathbb{I}_r(\mathrm{t}) \end{cases}, \quad \mathrm{J_0(t)} = \begin{cases} S_{h0}(\mathrm{t}) \\ \mathbb{E}_{h0}(\mathrm{t}) \\ \mathbb{I}_{h0}(\mathrm{t}) \\ \mathbb{Q}_{h0}(\mathrm{t}) \\ \mathbb{R}_{h0}(\mathrm{t}) \\ S_{r0}(\mathrm{t}) \\ \mathbb{E}_{r0}(\mathrm{t}) \\ \mathbb{I}_{r0}(\mathrm{t}) \end{cases},
$$

$$
\mathrm{B}(t, \mathrm{J(t)}) = \begin{cases} \mathrm{B}_1(t, S_h, \mathbb{E}_h, \mathbb{I}, \mathbb{Q}, \mathbb{R}_h, S_r, \mathbb{E}_r, \mathbb{I}_r) \\ \mathrm{B}_2(t, S_h, \mathbb{E}_h, \mathbb{I}, \mathbb{Q}, \mathbb{R}_h, S_r, \mathbb{E}_r, \mathbb{I}_r) \\ \mathrm{B}_3(t, S_h, \mathbb{E}_h, \mathbb{I}, \mathbb{Q}, \mathbb{R}_h, S_r, \mathbb{E}_r, \mathbb{I}_r) \\ \mathrm{B}_4(t, S_h, \mathbb{E}_h, \mathbb{I}, \mathbb{Q}, \mathbb{R}_h, S_r, \mathbb{E}_r, \mathbb{I}_r) \\ \mathrm{B}_5(t, S_h, \mathbb{E}_h, \mathbb{I}, \mathbb{Q}, \mathbb{R}_h, S_r, \mathbb{E}_r, \mathbb{I}_r) \\ \mathrm{B}_6(t, S_h, \mathbb{E}_h, \mathbb{I}, \mathbb{Q}, \mathbb{R}_h, S_r, \mathbb{E}_r, \mathbb{I}_r) \\ \mathrm{B}_7(t, S_h, \mathbb{E}_h, \mathbb{I}, \mathbb{Q}, \mathbb{R}_h, S_r, \mathbb{E}_r, \mathbb{I}_r) \\ \mathrm{B}_8(t, S_h, \mathbb{E}_h, \mathbb{I}, \mathbb{Q}, \mathbb{R}_h, S_r, \mathbb{E}_r, \mathbb{I}_r) \end{cases} \tag{5}
$$

Applying the integral to both sides equation (4),

$$
\int_0^t \frac{d}{dt}\mathrm{J(t)}\, dt = \int_0^t \mathrm{B}(t, \mathrm{J(t)})\, dt,
$$

Finally, the solution is given by

$$
\mathrm{J(t)} = \mathrm{J}_0 + \int_0^t \mathrm{B}(t, \mathrm{J(t)})\, dt, \tag{6}
$$

The operator equation corresponding to equation (6) is given by

$$\mathcal{F}\mathrm{J}(\mathrm{t}) = \mathrm{J}_0 + \int_0^t \mathrm{B}(t, \mathrm{J}(\mathrm{t}))\, dt. \tag{7}$$

For the existence of a solution, assume the following two assumptions, will be holds.
$(\mathcal{L}_1)$ $\exists$ constants b^* and c^* (where b^* and c^* are positive) such that

$$|\mathrm{B}(t, \mathrm{J}(\mathrm{t}))| \leq b^*|\mathrm{J}(\mathrm{t})| + c^*,$$

$(\mathcal{L}_2)$ $\exists$ constant $k_p > 0$, for each J & $\mathrm{J}^* \in X$, such that

$$|\mathrm{B}(t, \mathrm{J}(\mathrm{t})) - \mathrm{B}(t, \mathrm{J}^*(\mathrm{t}))| \leq k_p|\mathrm{J} - \mathrm{J}^*|, \tag{8}$$

Define the Banach space $\mathcal{Z} = \mathcal{X} * \mathcal{X} * \mathcal{X} * \mathcal{X} * \mathcal{X} * \mathcal{X} * \mathcal{X} * \mathcal{X}$, such that $\mathcal{X} = \mathcal{C}[0, \mathcal{T}]$, and the Norm is defined by

$$||\mathcal{M}|| = \max_{t\in[0,\mathcal{T}]} ||S_h + E_h + I_h + Q_h + R_h + S_r + E_r + I_r||.$$

Theorem 4.1. If (L_1) and (L_2) hold, then operator equation (7) has at least one solution.
Proof: We have to the following operator equations

$$\mathcal{F}\mathrm{J}(\mathrm{t}) = \mathrm{J}_0 + \int_0^t \mathrm{B}(t, \mathrm{J}(\mathrm{t}))\, dt$$

and

$$\mathcal{F}\mathrm{J}^*(\mathrm{t}) = \mathrm{J}_0 + \int_0^t \mathrm{B}(t, \mathrm{J}^*(\mathrm{t}))\, dt$$

Consider that

$$\begin{aligned}\mathcal{F}\mathrm{J}(\mathrm{t}) - \mathcal{F}\mathrm{J}^*(\mathrm{t}) &= \mathrm{J}_0 + \int_0^t B(t, \mathrm{J}(\mathrm{t}))\, dt - \mathrm{J}_0 - \int_0^t B(t, \mathrm{J}^*(\mathrm{t}))\, dt,\\ &= \int_0^t \{\mathrm{B}(t, \mathrm{J}(\mathrm{t})) - \mathrm{B}(t, \mathrm{J}^*(\mathrm{t}))\}\, dt,\end{aligned}$$

That implies

$$\because |\mathrm{B}(t, \mathrm{J}(\mathrm{t})) - \mathrm{B}(t, \mathrm{J}^*(\mathrm{t}))| = k_p|\mathrm{J} - \mathrm{J}^*|$$

$$|\mathcal{F}\mathrm{J}(\mathrm{t}) - \mathcal{F}\mathrm{J}^*(\mathrm{t})| \leq \int_0^t k_p|\mathrm{J} - \mathrm{J}^*|\, dt,$$

Let $t = \max[0, T] = T$

$$|\mathcal{F}\mathrm{J}(\mathrm{t}) - \mathcal{F}\mathrm{J}^*(\mathrm{t})| \leq Tk_p|\mathrm{J} - \mathrm{J}^*|, \tag{9}$$

From equation (9), Let $Tk_p = k_l$

$$|\mathcal{F}\mathrm{J}(\mathrm{t}) - \mathcal{F}\mathrm{J}^*(\mathrm{t})| \leq k_l|\mathrm{J} - \mathrm{J}^*|. \tag{10}$$

Hence the operator equation satisfies Lipschitz Condition. Therefore, the solution system 4 is exist. Furthermore, have to show that $\mathcal{F}\mathrm{J}(\mathrm{t})$ is contraction. Now for uniform continuity, let $t_2 > t_1 \in [0, T]$

Consider that

$$\mathcal{F}\mathrm{J}(\mathrm{t}_2) - \mathcal{F}\mathrm{J}(\mathrm{t}_1) = \int_0^t \mathrm{B}(t_2, \mathrm{J}(t_2))\,dt - \int_0^t \mathrm{B}(t_1, \mathrm{J}(t_1))\,dt,$$

$$= \left|\{\mathrm{B}(t_2, \mathrm{J}(t_2)) - \mathrm{B}(t_1, \mathrm{J}(t_1))\}\right|. \tag{11}$$

As $t_1 < t_2$, from the right hand side of equation (11), become zero,

$$|\mathcal{F}\mathrm{J}(t_2) - \mathcal{F}\mathrm{J}(t_1)| \to 0, \text{ as } t_1 \to t_2 \tag{12}$$

it shows that $\mathrm{J}(\mathrm{t})$ is continuous.

As $\mathcal{F}\mathrm{J}(\mathrm{t})$ is bounded and continuous. Thus $\mathrm{J}(\mathrm{t})$ is uniformly continuous.

Hence by Arzela-Ascoli theorem, $\mathcal{F}\mathrm{J}(\mathrm{t})$ is relatively compact and completely continuous. Therefore the equation (7) has at least one solution.

Theorem 4.2. Using assumption (L_2), the operator equation (7) has a unique solution shows that the considered system (1) has a unique solution, if $k_l < 1$.

Proof: Let us define the operator

$$\mathcal{F}\mathrm{J}(\mathrm{t}) = \mathrm{J}_0(\mathrm{t}) + \int_0^t \mathrm{B}(t, \mathrm{J}(\mathrm{t}))\,dt,$$

for $\mathcal{F}\mathrm{J}, \mathcal{F}\mathrm{J}^* \in Z$, and Consider

$$|\mathcal{F}\mathrm{J}(\mathrm{t}) - \mathcal{F}\mathrm{J}^*(\mathrm{t})| = |\int_0^t \mathrm{B}(t, \mathrm{J}(\mathrm{t}))\,dt - \int_0^t \mathrm{B}(t, \mathrm{J}^*(\mathrm{t}))\,dt|,$$

$$= \int_0^t |(\mathrm{B}(t, \mathrm{J}(\mathrm{t})) - \mathrm{B}(t, \mathrm{J}^*(\mathrm{t}))\,dt|,$$

$$= |(\mathrm{B}(t, \mathrm{J}(\mathrm{t})) - \mathrm{B}(t, \mathrm{J}^*(\mathrm{t})))| \int_0^t dt.$$

From equation (8)

$$|\mathcal{F}\mathrm{J}(\mathrm{t}) - \mathcal{F}\mathrm{J}^*(\mathrm{t})| \le k_p|\mathrm{J}(\mathrm{t}) - \mathrm{J}^*(\mathrm{t})|,$$
$$\le k_l|\mathrm{J}(\mathrm{t}) - \mathrm{J}^*(\mathrm{t})|.$$

If $k_l < 1$, then the equation (7) has a unique solution and hence the equation (1) has a unique solution.

4.2. Stability Analysis

For stability of the consider problem, also assume small perturbation $\gamma \in C[0, T]$, such that, $\gamma(0) = 0$.

1. $|\gamma(\mathrm{t})| \le \epsilon \quad \text{for } \epsilon > 0,$
2. $\frac{d}{dt}\mathrm{J}(\mathrm{t}) = \mathrm{B}(t, \mathrm{J}(\mathrm{t})) + \gamma(\mathrm{t}), \quad \forall t \in [0, T].$

Lemma 4.1. The solution of the perturbed problem satisfies the following relation:

$$|\mathcal{F}\mathrm{J}(\mathrm{t}) - \mathcal{F}\mathrm{J}^*(\mathrm{t})| \le k_m$$

Proof: Solution of the perturbed problem is given by:

$$\mathcal{F}\mathrm{J}(\mathrm{t}) = \mathrm{J}_0(\mathrm{t}) + \int_0^t \mathrm{B}(t, \mathrm{J}(\mathrm{t}))\,dt + \int_0^t \gamma(\mathrm{t})\,dt,$$

For $\mathcal{F}\mathrm{J}^*(\mathrm{t})$ being any other solution and get the following
after simplification:

$$|\mathcal{F}\mathrm{J}(\mathrm{t}) - \mathcal{F}\mathrm{J}^*(\mathrm{t})| \le Tk_p|\mathrm{J} - \mathrm{J}^*|,$$
$$\le k_l|\mathrm{J} - \mathrm{J}^*|, \quad \because k_l = Tk_p$$
$$\le k_m, \quad \text{where } k_m = k_l|\mathrm{J} - \mathrm{J}^*|.$$

Theorem 4.3. In-view of equation (7) and Lemma (4.1), the solution of (4.2) is Ulam-Hyer (UH) stable and consequently the

model (1) is UH stable, if $\eta < 1$.
Proof: Consider

$$\begin{aligned}
|\mathcal{F}\mathrm{J}(\mathrm{t}) - \mathcal{F}\mathrm{J}^*(\mathrm{t})| &= |\mathrm{J}_0(\mathrm{t}) + \int_0^t \mathrm{B}(t, \mathrm{J}(\mathrm{t}))\, dt + \mathrm{J}_0^*(\mathrm{t}) - \int_0^t \mathrm{B}(t, \mathrm{J}^*(\mathrm{t}))\, dt|,\\
&\leq |\int_0^t \mathrm{B}(t, \mathrm{J}(\mathrm{t}))\, dt - \int_0^t \mathrm{B}(t, \mathrm{J}^*(\mathrm{t}))\, dt - \int_0^t \mathrm{B}(t, \mathrm{J}^*(\mathrm{t}))\, dt + \int_0^t \mathrm{B}(t, \mathrm{J}^*(\mathrm{t}))\, dt|\\
&+ |\int_0^t \gamma(\mathrm{t})\, dt + (-\int_0^t \gamma(\mathrm{t})\, dt)|,\\
|\mathcal{F}\mathrm{J}(\mathrm{t}) - \mathcal{F}\mathrm{J}^*(\mathrm{t})| &\leq |\int_0^t \mathrm{B}(t, \mathrm{J}(\mathrm{t}))\, dt - \int_0^t \mathrm{B}(t, \mathrm{J}^*(\mathrm{t}))\, dt - \int_0^t \mathrm{B}(t, \mathrm{J}^*(\mathrm{t}))\, dt + \int_0^t \mathrm{B}(t, \mathrm{J}^*(\mathrm{t}))\, dt|\\
&+ |\int_0^t \gamma(\mathrm{t})\, dt| + |(-\int_0^t \gamma(\mathrm{t})\, dt)|,\\
&\leq 2\{|\int_0^t |\mathrm{B}(t, \mathrm{J}(\mathrm{t})) - \mathrm{B}(t, \mathrm{J}^*(\mathrm{t}))|\}\, dt + \epsilon T + \epsilon T,\\
|\mathcal{F}\mathrm{J}(\mathrm{t}) - \mathcal{F}\mathrm{J}^*(\mathrm{t})| &\leq (2Tk_p)|\mathrm{J}(\mathrm{t}) - \mathrm{J}^*(\mathrm{t})| + 2\epsilon T,\\
|\mathrm{J}(\mathrm{t}) - \mathrm{J}^*(\mathrm{t})| &\leq \frac{2\epsilon T}{1 - 2Tk_p} \qquad \because Tk_p = k_l,\\
|\mathcal{F}\mathrm{J}(\mathrm{t}) - \mathcal{F}\mathrm{J}^*(\mathrm{t})| &\leq \frac{2\epsilon T}{1 - \omega}.
\end{aligned}$$

Hence, the solution of equation (7) is HU stable by using $Y_{A(\mathrm{t})} = \frac{2\epsilon T}{1-\omega}, Y(0) = 0$.

5. Numerical Investigation of Modified Monkey-Pox Disease Model

This section has been focusing on finding numerical solution for the proposed MP model (1). Applying NT on the first equation of the consider model (1),

$$\mathfrak{N}^+\{S_h{}'(t)\} = \mathfrak{N}^+[\Pi_h - \theta_1 I_r(\mathrm{t})S_h(\mathrm{t}) - \theta_2 I_h(\mathrm{t})S_h(\mathrm{t}) - \eta_h S_h(\mathrm{t})], \tag{13}$$

With the help of definition (2.6) the system (13),

$$\frac{s}{u}R(s,u) - \frac{S_h(0)}{u} = \mathfrak{N}^+[\Pi_h - \theta_1 I_r(\mathrm{t})S_h(\mathrm{t}) - \theta_2 I_h(\mathrm{t})S_h(\mathrm{t}) - \eta_h S_h(\mathrm{t})], \tag{14}$$

the nonlinear terms of proposed system (1),is expressed in the form of series as

$$\begin{aligned}
I_r(\mathrm{t})S_h(\mathrm{t}) &= \sum_{i=0}^{\infty} A_i(\mathrm{t}),\\
I_h(\mathrm{t})S_h(\mathrm{t}) &= \sum_{i=0}^{\infty} B_i(\mathrm{t}),\\
I_r(\mathrm{t})S_r(\mathrm{t}) &= \sum_{i=0}^{\infty} C_i(\mathrm{t}).
\end{aligned} \tag{15}$$

Where $A_i\,(t)$, $B_i\,(t)$ and $C_i\,(t)$ are the Adomian's polynomials defined as

$$A_i(\mathrm{t}) = \frac{1}{\Gamma(i+1)} \frac{\mathbf{d^i}}{d\boldsymbol{\lambda^i}} \left[\sum_{n=0}^{i} \boldsymbol{\lambda^n} I_{rn}(\mathrm{t}) \sum_{n=0}^{i} \boldsymbol{\lambda^n} S_{hn}(\mathrm{t}) \right]_{\lambda=0},$$
$$B_i(\mathrm{t}) = \frac{1}{\Gamma(i+1)} \frac{\mathbf{d^i}}{d\boldsymbol{\lambda^i}} \left[\sum_{n=0}^{i} \boldsymbol{\lambda^n} I_{hn}(\mathrm{t}) \sum_{n=0}^{i} \boldsymbol{\lambda^n} S_{hn}(\mathrm{t}) \right]_{\lambda=0},$$
$$C_i(\mathrm{t}) = \frac{1}{\Gamma(i+1)} \frac{\mathbf{d^i}}{d\boldsymbol{\lambda^i}} \left[\sum_{n=0}^{i} \boldsymbol{\lambda^n} I_{rn}(\mathrm{t}) \sum_{n=0}^{i} \boldsymbol{\lambda^n} S_{rn}(\mathrm{t}) \right]_{\lambda=0}. \tag{16}$$

By plugging (15) into (14), the following system is obtained

$$\mathfrak{N}^+ \left[\sum_{i=0}^{\infty} S_{hi}(t) \right] = \frac{S_h(0)}{s} + \frac{u}{s} \left[\mathfrak{N}^+ \{ \Pi_h - \theta_1 (\sum_{i=0}^{\infty} A_i(t)) - \theta_2 (\sum_{i=0}^{\infty} B_i(t)) - \eta_h (\sum_{i=0}^{\infty} S_{hi}(t)) \} \right],$$

By expanding the summations up-to three terms, the following series is obtained

$$\begin{aligned} \mathfrak{N}^+ \{ S_{h0}(t) + S_{h1}(t) + S_{h2}(t) + \dots \} &= \frac{S_h(0)}{s} + \frac{u}{s} [\mathfrak{N}^+ \{ \Pi_h - \theta_1 (A_0(t) + A_1(t) + A_2(t) \dots) \\ &- \theta_2 (B_0(t) + B_1(t) + B_2(t) + \dots) - \eta_h (S_{h0}(t) + S_{h1}(t) + S_{h2}(t) + \dots) \}]. \end{aligned} \tag{17}$$

Applying the inverse NT on equation (17),

$$S_h(0) = S_{h0}. \tag{18}$$

Similarly, for next iteration,

$$\mathfrak{N}^+ \{ S_{h1}(t) \} = \frac{u}{s} [\mathfrak{N}^+ \{ \Pi_h - \theta_1 A_0(t) - \theta_2 B_0(t) - \eta_h S_{h0}(t) \}]. \tag{19}$$

Now applying inverse NT on equation (19),

$$S_{h1}(\mathrm{t}) = \mathfrak{N}^{-1} [\frac{u}{s} [\mathfrak{N}^+ \{ \Pi_h - \theta_1 A_0(t) - \theta_2 B_0(t) - \eta_h S_{h0}(t) \}]. \tag{20}$$

Now to solve the equations (20), the values of $A_0(\mathrm{t})$, $A_1(\mathrm{t})$, $B_0(\mathrm{t})$, $B_1(\mathrm{t})$, $C_0(\mathrm{t})$ and $C_1(\mathrm{t})$, using the definition of Adomian polynomial,

$$A_i(\mathrm{t}) = \frac{1}{\Gamma(i+1)} \frac{\mathbf{d^i}}{d\boldsymbol{\lambda^i}} \left[\sum_{n=0}^{i} \boldsymbol{\lambda^n} I_{rn}(\mathrm{t}) \cdot \sum_{n=0}^{i} \boldsymbol{\lambda^n} S_{hn}(\mathrm{t}) \right]_{\lambda=0}. \tag{21}$$

By using i = 0 in equation (21),

$$A_0(\mathrm{t}) = I_{r0}(\mathrm{t}) S_{h0}(\mathrm{t}),$$

By using i = 1 in equation (21), equation (22) is obtained

$$A_1(\mathrm{t}) = \frac{\boldsymbol{d}}{d\lambda} \left[I_{r0}(\mathrm{t}) S_{h0}(\mathrm{t}) + \lambda I_{r0}(\mathrm{t}) S_{h1}(\mathrm{t}) + \dots + \lambda I_{r1}(\mathrm{t}) S_{h0}(\mathrm{t}) + \lambda^2 I_{r1}(\mathrm{t}) S_{h1}(\mathrm{t}) \right]_{\lambda=0}, \tag{22}$$

$$A_1(\mathrm{t}) = I_{r0}(\mathrm{t}) S_{h1}(\mathrm{t}) + I_{r1}(\mathrm{t}) S_{h0}(\mathrm{t}). \tag{23}$$

Similarly for $B(t)$ and $C(t)$

$$\begin{aligned}
&B_0(\mathrm{t}) = I_{h0}(\mathrm{t})S_{h0}(\mathrm{t}),\\
&B_1(\mathrm{t}) = I_{h0}(\mathrm{t})S_{h1}(\mathrm{t}) + I_{h1}(\mathrm{t})S_{h0}(\mathrm{t}),\\
&C_0(\mathrm{t}) = I_{r0}(\mathrm{t})S_{r0}(\mathrm{t}),\\
&C_1(\mathrm{t}) = I_{r0}(\mathrm{t})S_{r1}(\mathrm{t}) + I_{r1}(\mathrm{t})S_{r0}(\mathrm{t}).
\end{aligned} \tag{24}$$

Further, have need the following

$$\mathfrak{N}^{+}\{S_{h1}\,(t)\} \;=\; \frac{u}{s}[\,\mathfrak{N}^{+}\{\Pi_h - \theta_1 A_0\,(t) - \theta_2 B_0\,(t) \;-\eta_h\, S_{h0}\,(t)\}]. \tag{25}$$

Plugging the values $A_0\,(t),\;B_0\,(t)$ and $C_0\,(t)$ in equation (25),

$$\mathfrak{N}^{+}\{S_{h1}\,(t)\} = \;\frac{u}{s}[\,\mathfrak{N}^{+}\{\Pi_h - \theta_1 I_{r0}(\mathrm{t})S_{h0}(\mathrm{t}) - \theta_2 I_{h0}(\mathrm{t})S_{h0}(\mathrm{t}) - \eta_h\, S_{h0}\,(t)\}], \tag{26}$$

Using the definition (2.7),

$$\mathfrak{N}^{+}\{S_{h1}\,(t)\} \;=\; \frac{u}{s}[\,\mathfrak{N}^{+}(\Pi_h) - \theta_1\,\mathfrak{N}^{+}(I_{r0}\,(t)\,S_{h0}\,(t)) - \theta_2\mathfrak{N}^{+}(I_{h0}(\mathrm{t})S_{h0}(t)) - \eta_h\,\mathfrak{N}^{+}(S_{h0}\,(t))],$$

Using the same definition,

$$\mathfrak{N}^{+}\{S_{h1}\,(t)\} \;=\; \frac{u}{s}[\,(\frac{1}{s})\Pi_h - \theta_1(\frac{1}{s})(I_{r0}\,(t)\,S_{h0}\,(t)) - \theta_2(\frac{1}{s})(I_{h0}(\mathrm{t})S_{h0}(\mathrm{t})) - \eta_h\,(\frac{1}{s})(S_{h0}\,(t))], \tag{27}$$

Applying the inverse NT on equation (27),

$$S_{h1}(\mathrm{t}) = \boldsymbol{N}^{-1}\left[\frac{u}{\boldsymbol{s}^2}\right]\{\Pi_h - \theta_1(I_{r0}(\mathrm{t})S_{h0}(\mathrm{t})) - \theta_2(I_{h0}(\mathrm{t})S_{h0}(\mathrm{t})) - \eta_h(S_{h0}(\mathrm{t}))\}, \tag{28}$$

$$S_{h1}(\mathrm{t}) = t\{\Pi_h - \theta_1 I_{r0}(\mathrm{t})S_{h0}(\mathrm{t}) - \theta_2 I_{h0}(\mathrm{t})S_{h0}(\mathrm{t}) - \eta_h\;(S_{h0}(\mathrm{t})\},$$

For next iteration, have proceeds

$$\mathfrak{N}^{+}\{S_{h2}(\mathrm{t})\} = \frac{u}{s}\mathfrak{N}^{+}[\,\Pi_h - \;\theta_1\;A_1(\mathrm{t}) - \theta_2\;B_1(\mathrm{t}) - \eta_h S_{h1}(\mathrm{t})\,], \tag{29}$$

Using the $\mathrm{A}_1(\mathrm{t})$ and $\mathrm{B}_1(\mathrm{t})$,

$$\mathfrak{N}^{+}\{S_{h2}(\mathrm{t})\} = \frac{u}{s}\mathfrak{N}^{+}[\Pi_h - \;\theta_1(\;I_{r0}\,(t)\;\;S_{h1}\,(t)\;+I_{r1}(\mathrm{t})\;S_{h0}(\mathrm{t})\;) - \theta_2(I_{h0}(\mathrm{t})\;S_{h1}(\mathrm{t}) + I_{h1}(\mathrm{t})\;S_{h0}(\mathrm{t}))\;\;-\eta_h\;\;S_{h1}(\mathrm{t})],$$

$$\begin{aligned}
\mathfrak{N}^{+}\{S_{h2}(\mathrm{t})\} = \frac{u}{s}\mathfrak{N}^{+}\,[\,\Pi_h] - \frac{u}{s}\;\;[\theta_1\;I_{r0}\,(t)\;\;\;N^{+}\;\;(\;S_{h1}\,(t)) + \theta_1\;S_{h0}\,(t)\;\;\mathfrak{N}^{+}\,(\;I_{r1}\;\;(t)\;\;) + \theta_2\;I_{h0}\,(t)\\
\mathfrak{N}^{+}\;(\;S_{h1}\,(t)\;\;) + \;\theta_2\;S_{h0}\,(t)\;\;\mathfrak{N}^{+}\,(I_{h1}\,(t)) + \eta_h\mathfrak{N}^{+}\;(S_{h1}\,(t)\;\;)],
\end{aligned} \tag{30}$$

using the calculated values of $S_{h1}(\mathrm{t})$, $I_{h1}(\mathrm{t})$, $I_{r1}(\mathrm{t})$ in equation (30),

$$\mathfrak{N}^{+}\{S_{h2}(\mathrm{t})\} = \frac{u}{s}\left[\mathfrak{N}^{+}(\Pi_h)\right] - \frac{u}{s}\ [\theta_1\ I_{r0}(\mathrm{t})\,\mathfrak{N}^{+}[t\{\Pi_h - \theta_1 I_{r0}\,(t)\,S_{h0}\,(t) + \theta_2 I_{h0}\,(t)\,S_{h0}\,(t) - \eta_h\ (S_{h0}\,(t)\}] + \theta_1\ S_{h0}(\mathrm{t})\mathfrak{N}^{+}[t\{\xi_r E_{r0}\,(t) - \eta_r I_{r0}\,(t)\}] + \theta_2\ I_{h0}(\mathrm{t})\mathfrak{N}^{+}[t\{\Pi_h - \theta_1 I_{r0}\,(t)\,S_{h0}\,(t) + \theta_2 I_{h0}\,(t)\,S_{h0}\,(t) - \eta_h\ (S_{h0}\,(t)\}] + \theta_2\ S_{h0}\ \ (\mathrm{t})\mathfrak{N}^{+}\ [t\{\Pi_h - \theta_1 I_{r0}\,(t)\,S_{h0}\,(t) + \theta_2 I_{h0}\,(t)\,S_{h0}\,(t) - \eta_h\ (S_{h0}\,(t)\}] + \eta_h\ \mathfrak{N}^{+}\ [t\{\Pi_h - \theta_1 I_{r0}\,(t)\ S_{h0}\,(t) + \theta_2 I_{h0}\,(t)\,S_{h0}\,(t) - \eta_h\ (S_{h0}\,(t)\}],$$

Using the same definition,

$$\mathfrak{N}^{+}\{S_{h2}(\mathrm{t})\} = \frac{u}{S^2}\,[\Pi_h\,] - \frac{u^2}{S^3}[\theta_1\ \ I_{r0}(\mathrm{t})\ \{\Pi_h - \theta_1 I_{r0}\,(t)\,S_{h0}\,(t) + \theta_2 I_{h0}\,(t)\,S_{h0}\,(t) - \eta_h\ \ (S_{h0}\,(t)\} + \theta_1\ S_{h0}\,(t)\,\{\xi_r E_{r0}(t) - \eta_r I_{r0}\,(t)\} + \theta_2\ \ - I_{h0}(\mathrm{t})\ \{\Pi_h \theta_1 I_{r0}\,(t)\,S_{h0}\,(t) + \theta_2 I_{h0}\,(t)\ S_{h0}\,(t) - \eta_h\ (S_{h0}\,(t)\}] + \theta_2\ S_{h0}\ (\mathrm{t})\{\Pi_h - \theta_1 \tag{31}$$

$$I_{r0}\,(t)\,S_{h0}\,(t) + \theta_2 I_{h0}\,(t)\,S_{h0}\,(t) - \eta_h\ S_{h0}\,(t)\} + \eta_h\ \{\Pi_h - \theta_1 I_{r0}\,(t)\,S_{h0}\,(t) + \theta_2 I_{h0}\,(t)\,S_{h0}\,(t) - \eta_h\ (S_{h0}\,(t)\}], \tag{32}$$

Applying the inverse NT on equation (31),

$$S_{h2}(\mathrm{t}) = \mathfrak{N}^{-1}(\frac{u}{S^2})\ \ [\Pi_h\,] - \mathfrak{N}^{-1}(\frac{u^2}{S^3}\,)[\theta_1\ \ I_{r0}(\mathrm{t})\ \{\Pi_h - \theta_1 I_{r0}\,(t)\,S_{h0}\,(t) + \theta_2 I_{h0}\,(t)\,S_{h0}\,(t) - \eta_h\ \ (S_{h0}\,(t)\} + \theta_1\ S_{h0}\,(t)\,\{\xi_r E_{r0}\,(t) - \eta_r I_{r0}\,(t)\} + \theta_2\ I_{h0}(\mathrm{t})\ \{\Pi_h - \theta_1 I_{r0}\,(t)\,S_{h0}\,(t) + \theta_2 I_{h0}\,(t)\,S_{h0}\,(t) - \eta_h\ (S_{h0}\,(t)\}] + \theta_2\ S_{h0}\ (\mathrm{t})\{\Pi_h - \theta_1 I_{r0}\,(t)\,S_{h0}\,(t) + \theta_2 I_{h0}\,(t)\,S_{h0}\,(t) - \eta_h\ S_{h0}\,(t)\} + \eta_h\ \{\Pi_h - \theta_1 I_{r0}\,(t)\,S_{h0}\,(t) + \theta_2 I_{h0}\,(t)\,S_{h0}\,(t) - \eta_h\ (S_{h0}\,(t)\}], \tag{33}$$

By simple computational,

$$S_{h2}(\mathrm{t}) = t\,[\Pi_h] - \frac{t^2}{2}\ [\theta_1\ \ I_{r0}(\mathrm{t})\ \{\Pi_h - \theta_1 I_{r0}\,(t)\,S_{h0}\,(t) + \theta_2 I_{h0}\,(t)\,S_{h0}\,(t) - \eta_h\ \ (S_{h0}\,(t)\} + \theta_1\ S_{h0}\,(t)\,\{\xi_r E_{r0}\,(t) - \eta_r I_{r0}\,(t)\} + \theta_2\ I_{h0}(\mathrm{t})\ \{\Pi_h\,(t) - \theta_1 I_{r0}\,(t)\,S_{h0} + \theta_2 I_{h0}\,(t)\,S_{h0}\,(t) - \eta_h\ (S_{h0}\,(t)\}] + \theta_2\ S_{h0}\ (\mathrm{t})\{\Pi_h\,(t) - \theta_1 I_{r0}\,(t)\,S_{h0} + \theta_2 I_{h0}\,(t)\,S_{h0}\,(t) - \eta_h\ S_{h0}\,(t)\} + \eta_h\ \{\Pi_h - \theta_1 I_{r0}\,(t)\ S_{h0}\,(t) + \theta_2 I_{h0}\,(t)\,S_{h0}\,(t) - \eta_h\ (S_{h0}\,(t)\}]. \tag{34}$$

In similar way for the other compartments, have get the solution that are given below,

$$\begin{aligned} &E_h(0) = E_{h0}, \\ &E_{h1}(\mathrm{t}) = t\{\theta_1 I_{r0}(\mathrm{t}) S_{h0}(\mathrm{t}) + \theta_2 I_{h0}(\mathrm{t}) S_{h0}(\mathrm{t}) - (\tau_1 + \tau_2 + \eta_h + \alpha_1) E_{h0}(\mathrm{t})\}, \\ &E_{h2}\ (\mathrm{t}) = \frac{t^2}{2}\Big[\theta_1 I_{r0}(\mathrm{t})\{\Pi_h - \theta_1 I_{r0}(\mathrm{t}) S_{h0}(\mathrm{t}) + \theta_2 I_{h0}(\mathrm{t}) S_{h0}(\mathrm{t}) - \eta_h(S_{h0}(\mathrm{t}))\} \\ &\quad + \theta_1 S_h(0)\{\xi_r E_{r0}(\mathrm{t}) - \eta_r I_{r0}(\mathrm{t})\} + \theta_2 I_h(0)\mathfrak{N}^{+}\{t\}\{\Pi_h - \theta_1 I_{r0}(\mathrm{t}) \\ &\quad S_{h0}(\mathrm{t}) + \theta_2 I_{h0}(\mathrm{t}) S_{h0}(\mathrm{t}) - \eta_h(S_{h0}(\mathrm{t}))\} + \theta_2 S_h(0)\{\tau_1 E_{h0}(\mathrm{t}) - (\ \psi\ + \eta_h \\ &\quad + \rho_1 + \alpha_2\ ) I_{h0}(\mathrm{t}) - (\ \tau_1 + \tau_2 +\ \ \eta_h + \alpha_1\ )\ \{\theta_1 I_{r0}(\mathrm{t}) S_{h0}(\mathrm{t}) + \theta_2 I_{h0}(\mathrm{t}) S_{h0}(\mathrm{t}) \\ &\quad - (\ \tau_1 + \tau_2 + \eta_h + \alpha_1) E_{h0}(\mathrm{t})\}\Big]. \end{aligned} \tag{35}$$

In similar method for the I_h

$$\begin{aligned} &I_h(0) = I_{h0}, \\ &I_{h1}(\mathrm{t}) = t\,\{\tau_1 E_{h0}(\mathrm{t}) -\ (\psi + \eta_h +\ \ \rho_1 + \alpha_2\ )\ I_{h0}(\mathrm{t})\}\,, \\ &I_{h2}\ (\mathrm{t}) = \frac{t^2}{2}\Big[\tau_1\ \{\ t\}\{\theta_1 I_{r0}\,(t)\,S_{h0}\,(t) + \theta_2 I_{h0}(\mathrm{t}) S_{h0}(\mathrm{t}) - (\ \tau_1 + \tau_2 + \eta_h + \alpha_1) E_{h0}\,(t)\} - (\ \psi\ + \eta_h \\ &\quad + \rho_1 + \alpha_2)\{\tau_1 E_{h0}\,(t)\ - (\ \psi\ + \eta_h +\ \ \rho_1 + \alpha_2) I_{h0}\,(t)\}\Big]. \end{aligned} \tag{36}$$

In similar method for the Q_h,

$$\begin{aligned} &Q_h(0) = Q_{h0}, \\ &Q_{h1}(\mathrm{t}) = t\{\tau_2 E_{h0}(\mathrm{t}) - (\phi + \rho_2 + \eta_h + \alpha_3) Q_{h0}(\mathrm{t})\}, \\ Q_{h2}(\mathrm{t}) = &\frac{t^2}{2}[\tau_2\{\theta_1 I_{r0}(t) S_{h0}(t) \quad + \theta_2 I_{h0}(\mathrm{t}) S_{h0}(t) - \ \tau_1 + \tau_2 + \ \eta_h + \alpha_1)\, E_{h0}(t)\} \\ &- (\phi + \rho_2 + \eta_h + \alpha_3\ )\ \{\tau_2 E_{h0}(t)\ - (\phi + \rho_2 + \eta_h + \alpha_3\ )\, Q_{h0}(t)\}]. \end{aligned} \tag{37}$$

On the same way, the author can find R_h as

$$\begin{aligned} &R_h(0) = R_{h0}, \\ &R_{h1}(\mathrm{t}) = t\{\psi I_{h0}(\mathrm{t}) + \phi Q_{h0}(\mathrm{t}) - \eta_h R_{h0}(\mathrm{t})\}, \\ &R_{h2}\ (\mathrm{t}) = \frac{t^2}{2}\ [\psi\,\{\tau_1 E_{h0}(t) - (\ \psi\ + \eta_h +\ \rho_1 + \alpha_2\ )\ I_{h0}(t)\} + \phi\,\{\tau_2 E_{h0}(t) - (\phi + \rho_2 + \eta_h + \alpha_3\ )\ Q_{h0}(t)\} \\ &\quad - \eta_h\,\{\psi I_{h0}(t) + \phi Q_{h0}(t) - \eta_h R_{h0}(t)\}]. \end{aligned} \tag{38}$$

Flowing the same method for the S_r,

$$\begin{aligned} &S_r(0) = S_{r0}, \\ &S_{r1}(\mathrm{t}) = t\{\Pi_r - \theta_r I_{r0}(\mathrm{t}) S_{r0}(\mathrm{t}) - \eta_r S_{r0}(\mathrm{t})\}, \\ &S_{r2}(t) = t\left\{\Pi_r\ \right\} - \frac{t^2}{2}\left[\theta_r I_{r0}(t)\{\Pi_r -\ \theta_r I_{r0}(t) S_{r0}(t) - \eta_r S_{r0}(t)\} + \theta_r S_{r0}(t)\left\{\xi_r E_{r0}(t) - \eta_r I_{r0}(t)\}\right]\right. \\ &\quad - \eta_r\{\Pi_r -\ \theta_r I_{r0}(t) S_{r0}(t) - \eta_r S_{r0}(t)\}. \end{aligned}$$

One can always E_r as

$$\begin{aligned} &E_r(0) = E_{r0}, \\ &E_{r1}(\mathrm{t}) = t\{\theta_r I_{r0}(\mathrm{t}) S_{r0}(\mathrm{t}) - (\eta_r + \xi_r) E_{r0}(\mathrm{t})\}, \end{aligned}$$

$$\begin{aligned} E_{r2}(\mathrm{t}) = &\frac{t^2}{2}\,[\theta_r I_{r0}(t)\,\{\Pi_r -\ \theta_r I_{r0}(t)\, S_{r0}(t) - \eta_r S_{r0}(t)\}\ + \theta_r S_{r0}(t)\,\{\xi_r E_{r0}(t) - \eta_r I_{r0}(t)\} - (\eta_r + \xi_r) \\ &\{\theta_r I_{r0}(t)\, S_{r0}(t) - (\eta_r + \xi_r) E_{r0}(t)\}]. \end{aligned} \tag{39}$$

In similar method for the I_r,

$$\begin{aligned} &I_r(0) = I_{r0}, \\ &I_{r1}(\mathrm{t}) = t\{\xi_r E_{r0}(\mathrm{t}) - \eta_r I_{r0}(\mathrm{t})\}, \\ &I_{r2}\,(\mathrm{t}) = \frac{t^2}{2}\,[\ \xi_r\{\theta_r I_{r0}(t)\, S_{r0}(t) - (\eta_r + \xi) E_{r0}(t)\}\ - \eta_r\{\xi_r E_{r0}(t) - \eta_r I_{r0}(t)\}]. \end{aligned} \tag{40}$$

6. Numerical Simulation

In the last section of the research work,the author provides the graphical analysis for the result obtained via proposed techniques, by some known computer software Mathematica. The simulation results were generated using various range for time parameters. The “Figure 5” and “Figure 10” indicate that as time goes on, more people get infected, if they receive more vaccinations.

Our numerical simulation has been focused on the real data, in the United State of America (USA) from May 15, 2023 to March 15, 2023.

6.1. Results

Now to discuss graphically behavior of different dynamical quantities as shown in the from “Figure 1” to “Figure 11”. With respect to α_1, α_2 and α_3, have consider different value as 0.03, 0.04 and 0.05, increase the value the dynamical quantities $S_h(\mathrm{t}), E_h t, I_h(\mathrm{t}), Q_h(\mathrm{t}), R_h(\mathrm{t})$ and $S_r(\mathrm{t}), E_r(\mathrm{t}), I_r(\mathrm{t})$ as shown in “Figure 1” to “Figure 11”, with respect to different range of time parameters.

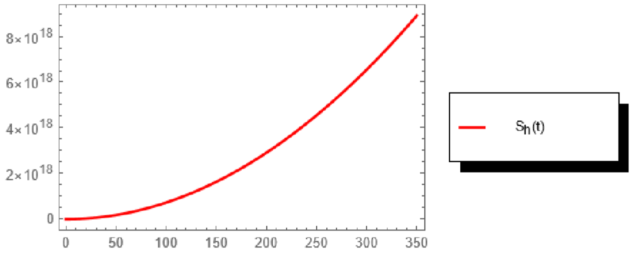

Figure 1. *This graph shows the dynamical behavior of* S_h.

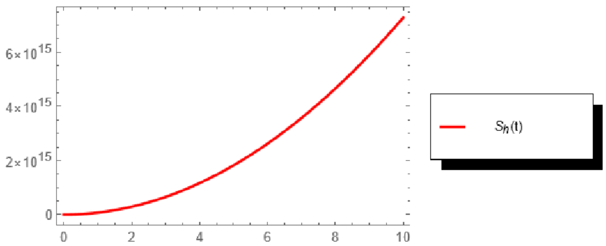

Figure 2. *This graph shows the dynamical behavior of* I_h.

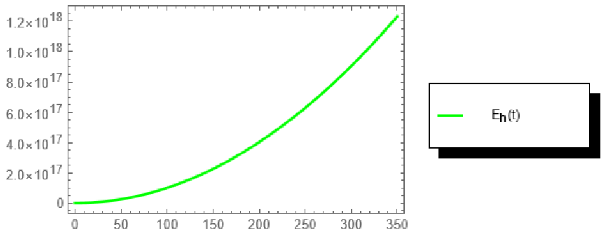

Figure 3. *This graph shows the dynamical behavior of* E_h.

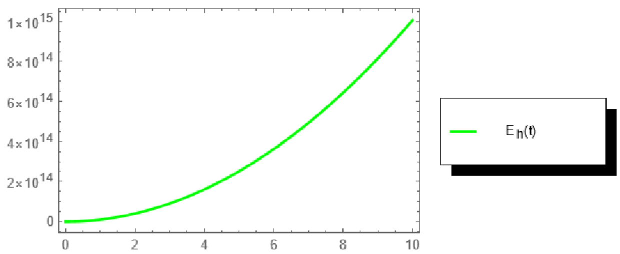

Figure 4. *This graph shows the dynamical behavior of* E_h.

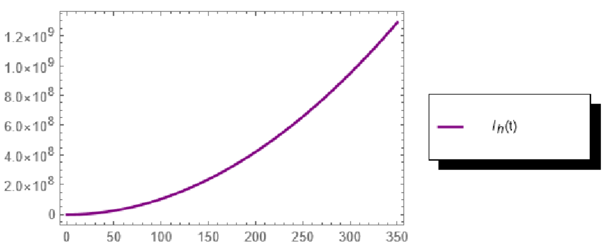

Figure 5. *This graph shows the dynamical behavior of* I_h.

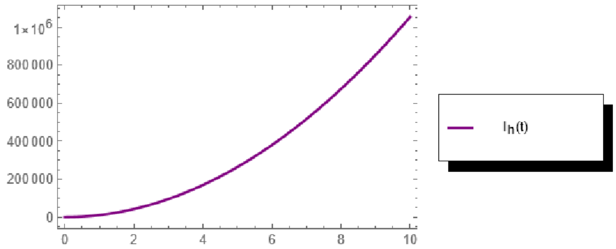

Figure 6. *This graph shows the dynamical behavior of* I_h.

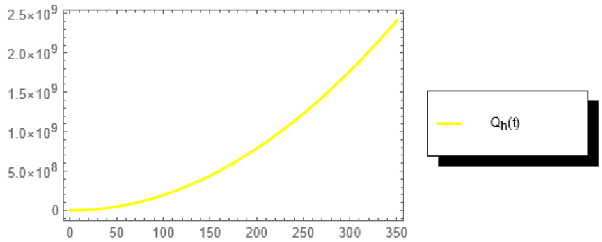

Figure 7. *This graph shows the dynamical behavior of* Q_h.

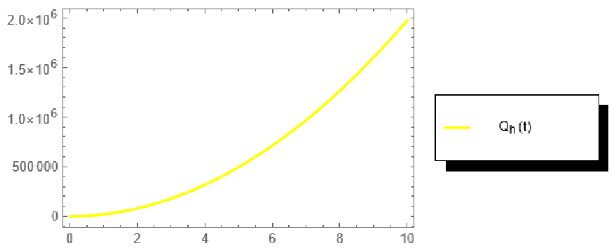

Figure 8. *This graph shows the dynamical behavior of* Q_h.

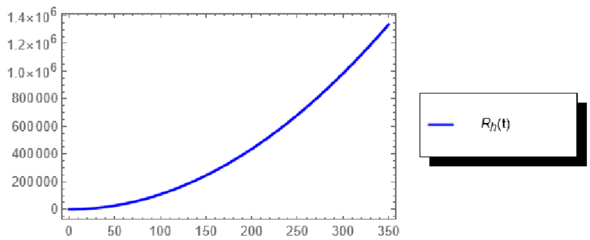

Figure 9. *This graph shows the dynamical behavior of* R_h.

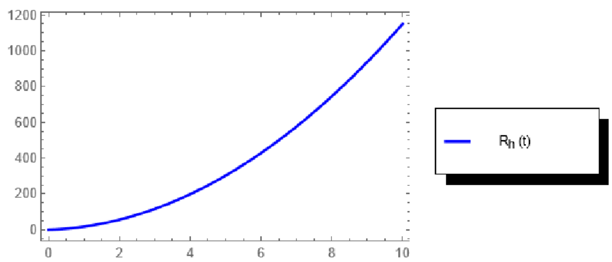

Figure 10. *This graph shows the dynamical behavior of* R_h.

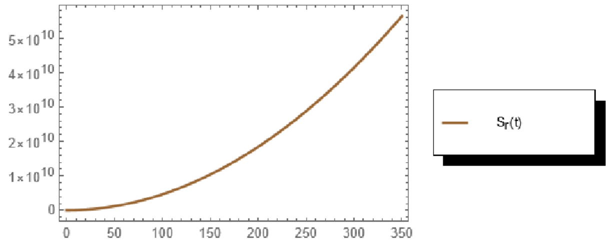

Figure 11. *This graph shows the dynamical behavior of* S_r.

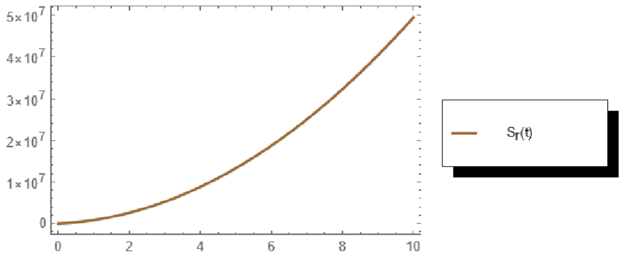

Figure 12. *This graph shows the dynamical behavior of* S_r.

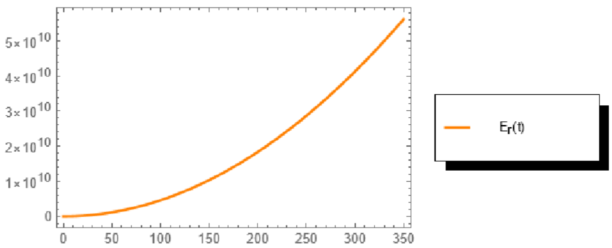

Figure 13. *This graph shows the dynamical behavior of* E_r.

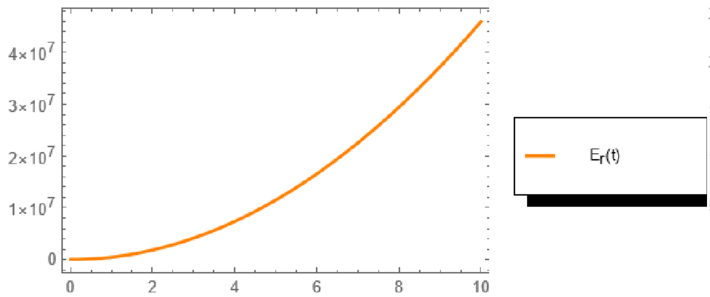

Figure 14. *This graph shows the dynamical behavior of* E_r.

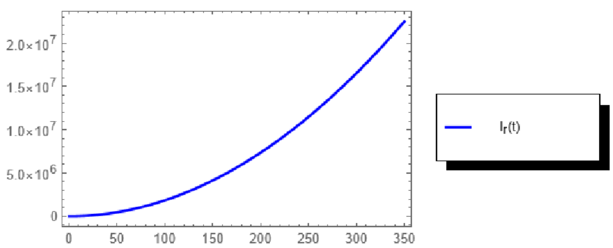

Figure 15. *This graph shows the dynamical behavior of* I_r.

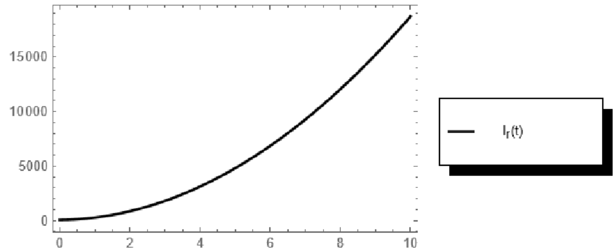

Figure 16. *This graph shows the dynamical behavior of* I_r.

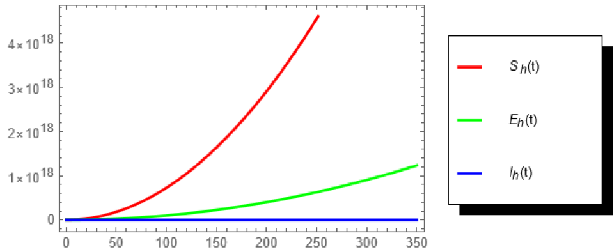

Figure 17. *This graph shows the dynamical behavior of* S_h, E_h, I_h.

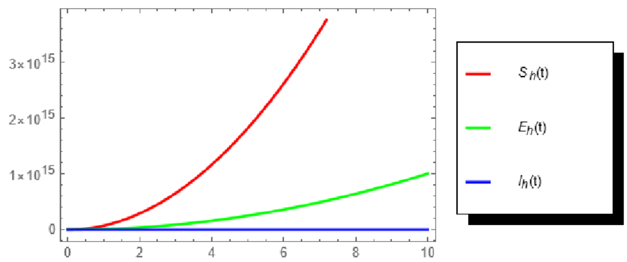

Figure 18. *This graph shows the dynamical behavior of* S_h, E_h, I_h.

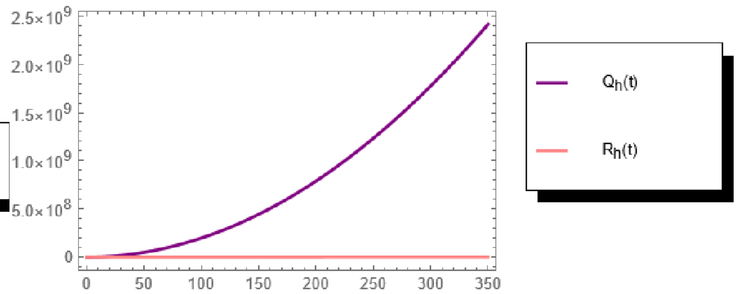

Figure 19. *This graph shows the dynamical behavior of* Q_h, R_h.

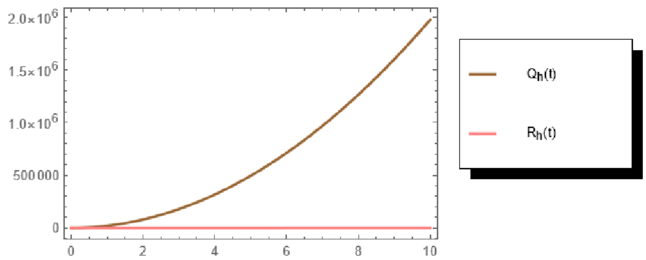

Figure 20. *This graph shows the dynamical behavior of* Q_h, R_h.

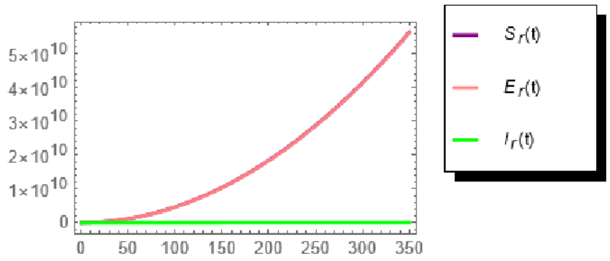

Figure 21. *This graph shows the dynamical behavior of* S_r, E_r, I_r.

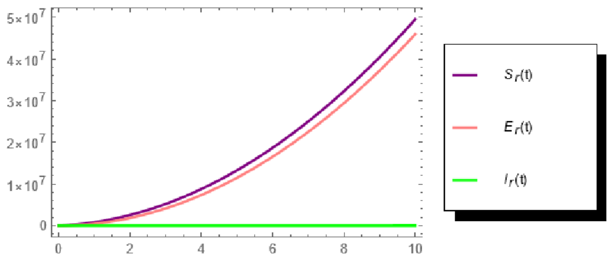

Figure 22. *This graph shows the dynamical behavior of* S_r, E_r, I_r.

6.2. Discussion

The given tables represent various captions. “Table 1” illustrates different compartments, while “Table 2” explains the various rates of parameters involved in the MP model. “Table 3” are consider vales for different compartments, and “Table 4” details the values for different compartments..

Table 4. *Initially considered values for different compartments.*

Compartments	Values	Compartments	Values
$S_h(t)$	333,279,356	$E_h(t)$	8000
$I_h(t)$	01	$Q_h(t)$	200
$R_h(t)$	0	$S_r(t)$	900
$E_r(t)$	400	$I_r(t)$	50

Table 5. *Parameter Values.*

Parameters	Values	Parameters	Values
π_h	11951.787886	τ_1	0.0306
η_h	0.0000358603	τ_2	0.0571
θ_1	0.000041552	ϕ	0.02951
θ_2	0.6307	ψ	0.0369
ρ_1	0.3356	ρ_2	0.04149
Π_r	182,623.318	η_r	0.0005479425
θ_r	0.1028	ξ_r	0.0799
α_1	0.03	α_2	0.04
α_3	0.05		

7. Conclusion

In the context of this computational investigation,the author investigated proposed mathematical model of MP via NADM. Increasing our understanding of the basic principles that govern the transmission of MP within a population was the major goal of this study. In addition to exposing remarkable discoveries, our findings provide an explanation for new understandings. A more accurate description of real-world events is provided by this study that clarifies important components such as the NADM is known to be essential in providing a detailed knowledge of the behavior of the system. Not only do the numerical simulations show there are solutions to the MP mathematical model, but they also offer important insights into the degree to that the system is sensitive to a variety of conditions. The findings presented here offer a significant contribution to the broader field of mention relevant field or discipline, since they provide both theoretical advances and practical consequences. The ideas that have been offered throughout this article have the potential to direct future research activities, hence influencing the creation of better models and approaches to intervention in the fight against MP. When the MP mathematical model and the NADM are combined, a powerful tool is produced that may be used to understand the intricate workings of infectious disease dynamics. This study not only broadens, understanding of MP, but it also highlights the significance of advanced mathematical techniques, such as artificial intelligence, in the context of addressing difficulties related to public health.

Abbreviation

ODEs	Ordinary Differential Equations
DEs	Differential Equations
MM	Mathematical Modeling
MP	Monkey-pox
MPV	Monkeypox Virus
NADM	Natural Adomain Decomposition Method

Author’s Contribution

Amjad Ali: Played a role in research article, generating new problem ideas, devising problem-solving methods, and reviewing literature.

Imtiazur Rahman: Applied Numerical techniques, contributed to solving the main problem and Responsible for writing, reviewing, editing, and submitting the initial draft.

Furqan Habib: Provided the core problem idea and contributed to draft review.

All authors contributed with equal enthusiasm and dedication.

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Availability of Data and Materials

Data related to this research article can be requested from the corresponding author.

Ethical Approval

This research project does not involve any form of gender discrimination, religious biases, or caste-related considerations.

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Conflicts of Interest

There is no competing interest by the authors to declare.

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