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# A Dynamic Frictionless Contact Problem with Adhesion in Thermo-elasto-viscoplasticity

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**Abstract:** The present paper is devoted to the study a dynamic problem describing a frictionless contact between a thermo-elasto-viscoplastic body and an adhesive foundation. The constitutive law includes a temperature effect described by the first order evolution equation. The contact is modelled with a normal compliance condition involving adhesion effect of contact surfaces. The adhesion is modelled with a surface variable, the bonding field whose evolution is described by a first order differential equation. A variational formulation for the problem is given as a system involving the displacement field, the bonding field and the temperature field. The existence and the uniqueness of the weak solution are established. The proof is based on evolution equation with monotone operators, differential equations and fixed point theorem.

**Keywords:** Thermo-elasto-viscoplastic Materials, Dynamic Process, Frictionless Contact, Normal Compliance, Adhesion, Weak Solution, Ordinary Differential Equation, Evolution Equation, Fixed Point

## 1. Introduction

The adhesive contact between bodies, when a glue is added to keep the surfaces from relative motion, is receiving increasing attention in the mathematical literature. Basic modelling can be found in [9, 10]. Analysis of models for adhesive contact can be found in [4, 6, 8, 11, 24], and in the monograph [26] and references therein. Results on frictionless adhesive contact can be found in [5, 12, 13, 20, 23]. An application of theory of adhesive contact in the medical field of prosthetic limbs was considered in [18, 19]. The novelty in all the above papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by $\beta$, it describes the pointwise fractional density of active bonds on the contact surface, and sometimes referred to as the intensity of adhesion. Following [9, 10], the bonding field satisfies the restrictions $0 \leq \beta \leq 1$, when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active, when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion, when $0 < \beta < 1$ the adhesion is partial and only a fraction $\beta$ of the bonds is active. The reader is referred to the extensive bibliography on the subject in [16, 17, 22, 25, 27] . The aim of this paper consists on the study of a dynamic process of a frictionless contact between a thermo-elasto-viscoplastic body and an adhesive foundation. The temperature effect is included in the constitutive law and is descibed by a differential heat equation. The contact is modelled with a normale compliance condition involving adhesion effect of contact surfaces. The adhesion is modelled with a surface variable, the bonding field whose evolution is described by a first order differential equation. The model is formulated as a system involving the displacement field, the bonding field and the temperature field. A variational formulation for the model is derived. The existence and uniqueness of weak solution are proved. The novelty in this paper consists on the coupling of an elastic-viscoplastic material with thermal effect and a frictionless adhesive contact without adhesive wear. Such problems arise in industry and medical field.

The paper is organised as follows. Notations and some preliminaries are presented in section 2. The mechanical problem, the assumptions on the data and the variational formulation of the problem are presented in section 3. The main existence and uniqueness result Theorem 4.1 and its proof based on arguments of evolution equations with monotone operators and a fixed point argument are given in section 4.

# 2. Notation and Preliminaries

The notation and some preliminary material are presented in this short section. For details see, e.g., [7].

Denote by $S_d$ the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while $(.)$ and $|\ .\ |$ represent the inner product and the Euclidean norm on $S_d$ and $\mathbb{R}^d$, respectively. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a regular boundary $\Gamma$ and let $\nu$ denote the unit outer normal on $\Gamma$. Using the notation

$$H = L^2(\Omega)^d = \{\mathbf{u} = (u_i) \,/\, u_i \in L^2(\Omega)\ \},$$

$$\mathcal{H} = \{\sigma = (\sigma_{ij}) \,/\, \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\ \},$$

$$H_1 = \{\mathbf{u} = (u_i) \in H \,/\, \varepsilon(\mathbf{u}) \in \mathcal{H}\ \},$$

$$\mathcal{H}_1 = \{\sigma \in \mathcal{H} \,/\, Div\, \sigma \in H\},$$

where $\varepsilon : H_1 \to \mathcal{H}$ and $Div : \mathcal{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})),\quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}),\ Div\, \sigma = (\sigma_{ij,\,j}).$$

Here and below, the indices $i$ and $j$ run between 1 to $d$, the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable.

The spaces $H$, $\mathcal{H}$, $H_1$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\mathbf{u}, \mathbf{v})_H = \int_\Omega u_i v_i \, dx \qquad \forall\, \mathbf{u}, \mathbf{v} \in H,$$

$$(\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\, dx \qquad \forall\, \sigma, \tau \in \mathcal{H},$$

$$(\mathbf{u}, \mathbf{v})_{H_1} = (\mathbf{u}, \mathbf{v})_H + (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}} \qquad \forall\, \mathbf{u}, \mathbf{v} \in H_1,$$

$$(\sigma, \tau)_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}} + (Div\, \sigma, Div\, \tau)_H \ \ \forall\, \sigma, \tau \in \mathcal{H}_1.$$

The associated norms on the spaces $H$, $\mathcal{H}$, $H_1$ and $\mathcal{H}_1$ are denoted by $|\ .\ |_H$, $|\ .\ |_{\mathcal{H}}$, $|\ .\ |_{H_1}$ and $|\ .\ |_{\mathcal{H}_1}$, respectively.

Let $H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$ and let $\gamma : H_1 \to H_\Gamma$ be the trace map. For every element $\mathbf{v} \in H_1$, using the notation $\mathbf{v}$ to denote the trace $\gamma\mathbf{v}$ of $\mathbf{v}$ on $\Gamma$ and denote by $v_\nu$ and $\mathbf{v}_\tau$ the normal and the tangential components of $\mathbf{v}$ on the boundary $\Gamma$ given by

$$v_\nu = \mathbf{v}.\nu,\ \mathbf{v}_\tau = \mathbf{v} - v_\nu \nu. \tag{1}$$

Denote by $\sigma_\nu$ and $\sigma_\tau$ the normal and the tangential traces of a function $\sigma \in \mathcal{H}_1$, and recall that when $\sigma$ is a regular function then

$$\sigma_\nu = (\sigma\nu).\nu,\ \sigma_\tau = \sigma\nu - \sigma_\nu \nu, \tag{2}$$

and the following Green's formula holds:

$$(\sigma, \varepsilon(\mathbf{v}))_{\mathcal{H}} + (Div\, \sigma, \mathbf{v})_H = \int_\Gamma \sigma\nu\ .\ \mathbf{v}\, da \qquad \forall\, \mathbf{v} \in H_1. \tag{3}$$

Finally, for any real Hilbert space $X$, the classical notation for the spaces $L^p(0, T; X)$ and $W^{k,p}(0, T; X)$, where $1 \le p \le +\infty$ and $k \ge 1$. is used. Let $C(0, T; X)$ and $C^1(0, T; X)$ the space of continuous and continuously differentiable functions from $[0, T]$ to $X$, respectively, with the norms

$$|\ \mathbf{f}\ |_{C(0,T;X)} = \max_{t\in[0,T]} |\ \mathbf{f}(t)\ |_X,$$

$$|\ \mathbf{f}\ |_{C^1(0,T;X)} = \max_{t\in[0,T]} |\ \mathbf{f}(t)\ |_X + \max_{t\in[0,T]} |\ \dot{\mathbf{f}}(t)\ |_X,$$

respectively. Moreover, for a real number $r$, using $r_+$ to represent its positive part, that is $r_+ = \max\{0, r\}$. Finally, for the convenience of the reader, the following version of the classical theorem of Cauchy-Lipschitz (see, e.g., 28, p. 60) is given in the following result.

*Theorem* 2.1. Assume that $(X, |\ .\ |_X)$ is a real Banach space and $T > 0$. Let $F(t, .) : X \to X$ be an operator defined a.e. on $(0, T)$ satisfying the following conditions:

1. $\exists\ L_F > 0$ such that $|\ F(t, x) - F(t, y)\ |_X \le L_F\ |\ x - y\ |_X\ \forall\, x, y \in X$, a.e. $t \in (0, T)$.
2. $\exists\ p \ge 1$ such that $t \longmapsto F(t, x) \in L^p(0, T; X)$ $\forall\, x \in X$.

Then for any $x_0 \in X$, there exists a unique function $x \in W^{1,\,p}(0, T; X)$ such that

$$\dot{x}(t) = F(t, x(t)) \qquad \text{a.e. } t \in (0, T),$$

$$x(0) = x_0.$$

Theorem 2.1 will be used in section 4 to prove the unique solvability of the intermediate problem involving the bonding field.

Moreover, if $X_1$ and $X_2$ are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(.,.)_{X_1\times X_2}$.

# 3. Problem Statement and Variational Formulations

This section concerns the physical setting of the contact problem. A thermo-elasto-viscoplastic body which occupies the domain $\Omega \subset \mathbb{R}^d$ with the boundary $\Gamma$ divided into three disjoint measurable parts $\Gamma_1, \Gamma_2$ and $\Gamma_3$ such that $meas(\Gamma_1) > 0$. The time interval of interest is $[0, T]$ where $T > 0$. The body is clamped on $\Gamma_1$ and so the displacement field vanishes there. A volume force of density $\mathbf{f}_0$ acts in $\Omega\times\ (0, T)$ and surface tractions of density $\mathbf{f_2}$ act on $\Gamma_2\times\ (0, T)$. Assume that the body is in adhesive frictionless contact with an obstacle, the so called foundation, over the potential contact surface $\Gamma_3$. Moreover, the process is dynamic, and thus the inertial terms are included in the equation of motion. The material is assumed to behave according to the general thermo-elasto-viscoplastic constitutive law given by

$$\sigma\,(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{F}\varepsilon(\mathbf{u}(t)) + \int_0^t \mathcal{G}(\sigma(s){-}\mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)), \varepsilon(\mathbf{u}(s)), \theta(s))\; ds \tag{4}$$

where $\sigma$ denotes the stress tensor, $\mathbf{u}$ represents the displacement field, $\dot{\mathbf{u}}$ the velocity, $\varepsilon(\mathbf{u})$ is the small strain tensor and $\theta$ is the temperature field. Here $\mathcal{A}$ and $\mathcal{F}$ are nonlinear operators describing the purely viscous and the elastic properties of the material respectively, $\mathcal{G}$ is nonlinear operator which depends on the temperature and which describing the viscoplastic behaviour of the material. Using dots for derivatives with respect to the time variable $t$. It follows from (4) that at each time moment, the stress tensor $\sigma(t)$ is split into two parts: $\sigma(t) = \sigma^V(t) + \sigma^R(t)$, where $\sigma^V(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t))$ represents the purely viscous part of the stress whereas $\sigma^R(t)$ satisfies a rate-type thermo-elasto-viscoplastic relation

$$\sigma^R(t) = \mathcal{F}\varepsilon(\mathbf{u}(t)) + \int_0^t \mathcal{G}(\sigma^R(s), \varepsilon(\mathbf{u}(s)), \theta(s))\; ds. \tag{5}$$

When $\mathcal{G} = 0$, (4) reduces to the Kelvin-Voigt viscoelastic constitutive law given by

$$\sigma(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{F}\varepsilon(\mathbf{u}(t)). \tag{6}$$

The evolution of the temperature field $\theta$ is governed (see $[1, 7]$) by the heat equation, obtained from the conservation of energy, and defined by the following differential equation for the temperature

$$\dot{\theta} - div(K\nabla\theta) = r(\dot{\mathbf{u}}) + q \tag{7}$$

where $K = (k_{ij})$ represents the thermal conductivity tensor, $div(K\nabla\theta) = (k_{ij}\theta_{,i})_{,i}$, $q(t)$ the density of volume heat sources, and $r(\dot{\mathbf{u}}(t))$ a nonlinear function of the velocity. In [1], the following linear function was used

$$r(\dot{\mathbf{u}}(t)) = -c_{ij}\dot{u}_{i,j}(t).$$

The associated temperature boundary condition on $\Gamma_3$ is described by

$$k_{ij}\theta_{,i}n_j = -k_e(\theta - \theta_R) \, + h_\tau(\mid \dot{\mathbf{u}}_\tau \mid) \text{ on } \Gamma_3 \times (0,T)\,,$$

where $\theta_R$ is the temperature of the foundation, and $k_e$ is the heat exchange coefficient between the body and the obstacle, and $h_\tau : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$ is a given tangential function. Analysis of contact problems with thermal effect can be in $[3, 21]$.

Then, the classical formulation of the dynamic contact problem is the following.

*Problem P. Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, *a stress field* $\sigma : \Omega \times [0,T] \to S_d$, *a bonding field* $\beta : \Gamma_3 \times [0,T] \to \mathbb{R}$ *and a temperature field* $\theta : \Omega \times [0,T] \to \mathbb{R}_+$ *such that*

$$\sigma \, = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \mathcal{F}\varepsilon(\mathbf{u}) \, + \int_0^t \mathcal{G}(\sigma(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)), \varepsilon(\mathbf{u}(s)), \theta(s))\; ds \quad \text{in } \Omega \times (0,T)\,, \tag{8}$$

$$\rho\ddot{\mathbf{u}} = Div\,\sigma + \mathbf{f}_0 \text{ in } \Omega \times (0,T)\,, \tag{9}$$

$$\mathbf{u} \, = \mathbf{0} \quad \text{ on } \quad \Gamma_1 \times (0,T)\,, \tag{10}$$

$$\sigma\nu \, = \mathbf{f}_2 \quad \text{ on } \quad \Gamma_2 \times (0,T)\,, \tag{11}$$

$$\dot{\theta} - div(K\nabla\theta) = r(\dot{\mathbf{u}}) + q \text{ on } \quad \Omega \times (0,T)\,, \tag{12}$$

$$-k_{ij}\frac{\partial\theta}{\partial\mathbf{x}_i}n_j = k_e(\theta - \theta_R) - h_\tau(\mid \dot{\mathbf{u}}_\tau \mid) \text{ on } \quad \Gamma_3 \times (0,T)\,, \tag{13}$$

$$-\sigma_{\,\nu} \, = p_{\,\nu}(u_\nu) - \gamma_\nu\beta^2 R_{\,\nu}(u_\nu) \qquad \text{on} \quad \Gamma_3 \times (0,T)\,, \tag{14}$$

$$-\sigma_\tau = p_\tau(\beta)\mathbf{R}_\tau(\mathbf{u}_\tau) \text{ on } \quad \Gamma_3 \times (0,T)\,, \tag{15}$$

$$\dot{\beta} = -(\beta(\gamma_\nu(R_{\,\nu}(u_\nu))^2 + \gamma_\tau \mid \mathbf{R}_{\,\tau}(\mathbf{u}_\tau) \mid^2) - \varepsilon_a)_+ \text{ on } \quad \Gamma_3 \times (0,T)\,, \tag{16}$$

$$\theta = 0 \text{ on } \quad \Gamma_1 \cup \Gamma_2 \times (0,T)\,, \tag{17}$$

$$\mathbf{u}(0) = \mathbf{u}_0,\; \dot{\mathbf{u}}(0) = \mathbf{v}_0,\; \theta(0) = \theta_0 \; \text{ in } \; \Omega, \tag{18}$$

$$\beta(0) = \beta_0 \text{ on } \Gamma_3. \tag{19}$$

Here, (8) and (12) represent the thermo-elasto-viscoplastic constitutive law introduced in the third section, (9) represents the equation of motion where $\rho$ represents the mass density. (10)-(11) are the displacement-traction conditions. (13) represents the associated temperature boundary condition on $\Gamma_3$. Condition (14) represents the normal compliance conditions with adhesion where $\gamma_\nu$ is a given adhesion coefficient and $p_\nu$ is a given positive function which will be described below. In this condition the interpenetrability between the body and the foundation is allowed, that is $u_\nu$ can be positive on $\Gamma_3$. The contribution of the adhesive traction to the normal traction is represented by the term $\gamma_\nu\beta^2 R_\nu(u_\nu)$, the adhesive traction is tensile and is proportional, with proportionality coefficient $\gamma_\nu$, to the square of the intensity of adhesion and to the normal displacement, but as long as it does not exceed the bond length $L$. The maximal tensile traction is $\gamma_\nu L$. $R_\nu$ is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \le s \le 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The contact condition (14) was used in various papers, see e.g. $[4, 5, 24, 26]$. Condition (15) represents the adhesive contact condition on the tangential plane, in which $p_\tau$ is a given function and $\mathbf{R}_\tau$ is the truncation operator given by

$$\mathbf{R}_\tau(\mathbf{v}) = \begin{cases} \mathbf{v} & \text{if} \quad |\,\mathbf{v}\,| \le L, \\ L\frac{\mathbf{v}}{|\mathbf{v}|} & \text{if} \quad |\,\mathbf{v}\,| > L. \end{cases}$$

This condition shows that the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length $L$. The frictional tangential traction is assumed to be much smaller than the adhesive one and, therefore, omitted. The introduction of the operator $R_\nu$, together with the operator $\mathbf{R}_\tau$ defined above, is motivated by mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the parameter $L$ is made in what follows.

Next, the equation (16) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [4], see also $[24, 26]$ for more details. Here, besides $\gamma_\nu$, two new adhesion coefficients are involved, $\gamma_\tau$ and $\varepsilon_a$. Notice that in this model once debonding occurs bonding cannot be reestablished since, as it follows from (16), $\dot{\beta} \le 0$. (17) means that the temperature vanishes on $\Gamma_1 \cup \Gamma_2 \times (0, T)$. In (18) $\mathbf{u}_0$ is the given initial displacement, $\mathbf{v}_0$ is the given initial velocity and $\theta_0$ is the initial temperature. Finally, (19) represents the initial condition, in which $\beta_0$ denotes the initial bonding. To simplify the notation, the dependence of various functions on the variables $\mathbf{x} \in \Omega \cup \Gamma$ and $t \in [0, T]$ . is not indicated explicitely. To obtain the variational formulation of the problem (8)-(19), introducing for the bonding field the set

$$Z = \left\{\theta : [0, T] \to L^2(\Gamma_3) \;/\; 0 \le \theta(t) \le 1 \;\forall t \in [0, T]\,, \text{ a.e. on } \Gamma_3\right\}.$$

Let $E$ denote the closed subspace of $H^1(\Omega)$ given by

$$E = \left\{\gamma \in H^1(\Omega) \,/\, \gamma = 0 \text{ on } \Gamma_1 \cup \Gamma_2\right\}.$$

For the displacement field we need the closed subspace of $H_1$ defined by

$$V = \{\mathbf{v} \in H_1 \;/\; \mathbf{v} = 0 \text{ on } \Gamma_1\}.$$

Since $meas(\Gamma_1) > 0$, Korn's inequality holds (see [15]) and there exists a constant $C_k > 0$ which depends only on $\Omega$ and $\Gamma_1$ such that

$$|\,\varepsilon(\mathbf{v})\,|_{\mathcal{H}} \ge C_k \,|\,\mathbf{v}\,|_{H_1} \;\forall\, \mathbf{v} \in V.$$

The inner product and the associated norm on V are given by

$$(\mathbf{u}, \mathbf{v})_V = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \;\; |\,\mathbf{v}\,|_V = |\,\varepsilon(\mathbf{v})\,|_{\mathcal{H}} \;\; \forall \mathbf{u}, \mathbf{v} \in V. \tag{20}$$

It follows from Korn's inequality that $|\,.\,|_{H_1}$ and $|\,.\,|_V$ are equivalent norms on $V$ and therefore $(V, |\,.\,|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem there exists a constant $C_0$, depending only on $\Omega$, $\Gamma_1$ and $\Gamma_3$, such that

$$|\,\mathbf{v}\,|_{L^2(\Gamma_3)^d} \le C_0 \,|\,\mathbf{v}\,|_V \;\forall \mathbf{v} \in V. \tag{21}$$

In the study of the mechanical problem (8)-(19), assume the following assumptions. The viscosity operator $\mathcal{A} : \Omega \times S_d \to S_d$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_{\mathcal{A}} > 0 \text{ such that} \\ \mid \mathcal{A}(\mathbf{x},\varepsilon_1) - \mathcal{A}(\mathbf{x},\varepsilon_2) \mid \leq L_{\mathcal{A}} \mid \varepsilon_1 - \varepsilon_2 \mid \\ \quad \forall \varepsilon_1, \varepsilon_2 \in S_d, \text{a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ There exists a constant } m_{\mathcal{A}} > 0 \text{ such that} \\ (\mathcal{A}(\mathbf{x},\varepsilon_1) - \mathcal{A}(\mathbf{x},\varepsilon_2)).(\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{A}} \mid \varepsilon_1 - \varepsilon_2 \mid^2 \\ \quad \forall \varepsilon_1, \varepsilon_2 \in S_d, \text{ a.e. } \mathbf{x} \in \Omega. \\ (c) \text{ The mapping } \mathbf{x} \to \mathcal{A}(\mathbf{x},\varepsilon) \text{ is Lebesgue measurable} \\ \text{ on } \Omega \text{ for any } \varepsilon \in S_d. \\ (d) \text{ The mapping } \mathbf{x} \to \mathcal{A}(\mathbf{x},\mathbf{0}) \in \mathcal{H}. \end{cases} \tag{22}$$

The elasticity operator $\mathcal{F} : \Omega \times S_d \to S_d$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_{\mathcal{E}} > 0 \text{ such that} \\ \mid \mathcal{F}(\mathbf{x},\varepsilon_1) - \mathcal{F}(\mathbf{x},\varepsilon_2) \mid \leq L_{\mathcal{E}} \mid \varepsilon_1 - \varepsilon_2 \mid \\ \quad \forall \varepsilon_1, \varepsilon_2 \in S_d, \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ For any } \varepsilon \in S_d, \mathbf{x} \to \mathcal{F}(\mathbf{x},\varepsilon) \text{ is Lebesgue measurable} \\ \text{ on } \Omega. \\ (c) \text{ The mapping } \mathbf{x} \to \mathcal{F}(\mathbf{x},\mathbf{0}) \in \mathcal{H}. \end{cases} \tag{23}$$

The visco-plasticity operator $\mathcal{G} : \Omega \times S_d \times S_d \to S_d$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_{\mathcal{G}} > 0 \text{ such that} \\ \mid \mathcal{G}(\mathbf{x},\sigma_1,\varepsilon_1,\theta_1) - \mathcal{G}(\mathbf{x},\sigma_2,\varepsilon_2,\theta_2) \mid \\ \qquad \leq L_{\mathcal{G}}(\mid \sigma_1 - \sigma_2 \mid + \mid \varepsilon_1 - \varepsilon_2 \mid + \theta_1 - \theta_2 \mid) \\ \forall \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2, \in S_d, \theta_1, \theta_2 \in \mathbb{R}_+ \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ For any } \sigma, \varepsilon \in S_d, \theta \in \mathbb{R}_+ , \\ \quad \mathbf{x} \to \mathcal{G}(\mathbf{x},\sigma,\varepsilon,\theta) \text{ is Lebesgue measurable on } \Omega. \\ (c) \text{ The mapping } \mathbf{x} \to \mathcal{G}(\mathbf{x},\mathbf{0},\mathbf{0},0) \in \mathcal{H}. \end{cases} \tag{24}$$

The contact function $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_\nu > 0 \text{ such that} \\ \mid p_\nu(\mathbf{x},r_1) - p_\nu(\mathbf{x},r_2) \mid \leq L_\nu \mid r_1 - r_2 \mid \\ \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ (b) \text{ The mapping } \mathbf{x} \to p_\nu(\mathbf{x},r) \text{ is measurable on } \Gamma_3, \\ \text{ for any } r \in \mathbb{R}. \\ (c)\ p_\nu(\mathbf{x},r) = 0 \text{ for all } r \leq 0, \text{ a.e. } \mathbf{x} \in \Gamma_3. \end{cases} \tag{25}$$

The tangential contact function $p_\tau : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_\tau > 0 \text{ such that} \\ \mid p_\tau(\mathbf{x},d_1) - p_\tau(\mathbf{x},d_2) \mid \leq L_\tau \mid d_1 - d_2 \mid \\ \forall d_1, d_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ (b) \text{ There exists } M_\tau > 0 \text{ such that} \\ \mid p_\tau(\mathbf{x},d) \mid \leq M_\tau \ \forall d \in \mathbb{R}, \text{a.e. } \mathbf{x} \in \Gamma_3. \\ (c) \text{ The mapping } \mathbf{x} \to p_\tau(\mathbf{x},d) \text{ is measurable on } \Gamma_3, \\ \text{ for any } d \in \mathbb{R}. \\ (d) \text{ The mapping } \mathbf{x} \to p_\tau(\mathbf{x},0) \in L^2(\Gamma_3). \end{cases} \tag{26}$$

The tangential function $h_\tau : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_\tau > 0 \text{ such that} \\ \mid h_r(\mathbf{x},r_1) - h_r(\mathbf{x},r_2) \mid \leq L_\tau \mid r_1 - r_2 \mid \\ \forall r_1, r_2 \in \mathbb{R}_+, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ (b) \text{ The mapping } \mathbf{x} \to h_r(\mathbf{x},r) \in L^2(\Gamma_3) \\ \text{ is Lebesgue measurable on } \Gamma_3, \forall r \in \mathbb{R}_+. \end{cases} \tag{27}$$

The mass density satisfies

$$\rho \in L^\infty(\Omega), \text{ there exists } \rho^* > 0 \text{ such that } \rho(\mathbf{x}) \geq \rho^* \qquad \text{a.e. } \mathbf{x} \in \Omega. \tag{28}$$

The adhesion coefficient and the limit bound satisfy

$$\gamma_\nu, \gamma_\tau, \varepsilon_a \in L^\infty(\Gamma_3), \gamma_\nu \geq 0, \gamma_\tau \geq 0, \varepsilon_a \geq 0. \tag{29}$$

The body forces and surface traction have the regularity

$$\mathbf{f}_0 \in L^2(0,T;H)\ , \mathbf{f}_2 \in L^2(0,T;L^2(\Gamma_2)^d). \tag{30}$$

For the thermal tensors and the heat source density, suppose that

$$q \in L^2(0,T;L^2(\Omega)), \tag{31}$$

and for some $c_k > 0$, for all $(\xi_i) \in \mathbb{R}^d$ :

$$K = (k_{ij}), k_{ij} = k_{ji} \in L^\infty(\Omega), k_{ij}\xi_i\xi_j \geq c_k\xi_i\xi_j. \tag{32}$$

The boundary and initial data satisfy

$$\begin{aligned} &\mathbf{u}_0 \in V,\ \mathbf{v}_0 \in H,\ \theta_0 \in E, \\ &\theta_R \in L^2(0,T;L^2(\Gamma_3)),\ k_e \in L^\infty(\Omega, \mathbb{R}_+). \end{aligned} \tag{33}$$

$$\beta_0 \in L^2(\Gamma_3),\ 0 \leq \beta_0 \leq 1,\ \text{a.e. on } \Gamma_3. \tag{34}$$

The function $r : V \to L^2(\Omega)$ satisfies

$$\begin{cases} \text{There exists a constant } L_r > 0 \text{ such that} \\ \mid r(\mathbf{v}_1) - r(\mathbf{v}_2) \mid_{L^2(\Omega)} \leq L_r \mid \mathbf{v}_1 - \mathbf{v}_2 \mid\ \forall \mathbf{v}_1, \mathbf{v}_2 \in V. \end{cases} \tag{35}$$

Before giving the weak formulation of the mechanical problem, let us consider some concrete examples. A simple tangential function $h_\tau$ is given by

$$h_\tau(\mathbf{x}, r) = \lambda(\mathbf{x})r \quad \forall r \in \mathbb{R}_+,\ \text{a.e. } \mathbf{x} \in \Gamma_3,$$

where $\lambda \in L^\infty(\Gamma_3, \mathbb{R}_+)$ represents some rate coefficient for the gradient of the temperature. Using a modified inner product on $H = L^2(\Omega)^d$, given by

$$((\mathbf{u}, \mathbf{v}))_H = (\rho\mathbf{u}, \mathbf{v})_H \quad \forall \mathbf{u}, \mathbf{v} \in H,$$

that is, it is weighted with $\rho$, and let $\| \,.\, \|_H$ be the associated norm, i.e.,

$$\| \mathbf{v} \|_H = (\rho\mathbf{v}, \mathbf{v})_H^{\frac{1}{2}} \quad \forall \mathbf{v} \in H.$$

By assumption (28) $\| \,.\, \|_H$ and $| \,.\, |_H$ are equivalent norms on $H$. The embedding of $(V, | \,.\, |_V)$ into $(H, \| \,.\, \|_H)$ is continuous and dense. Denote by $V'$ the dual space of $V$. Identifying $H$ with its own dual, the Gelfand triiple is given by

$$V \subset H \subset V'.$$

Using the notation $(.,.)_{V' \times V}$ to represent the duality pairing between $V'$ and $V$, to have

$$(\mathbf{u}, \mathbf{v})_{V' \times V} = ((\mathbf{u}, \mathbf{v}))_H \quad \forall \mathbf{u} \in H, \forall \mathbf{v} \in V.$$

Assumptions (30) allow us, for a.e. $t \in (0,T)$, to define $\mathbf{f}(t) \in V'$ by

$$(\mathbf{f}(t), \mathbf{v})_{V' \times V} = \int_\Omega \mathbf{f}_0(t).\mathbf{v}\, dx + \int_{\Gamma_2} \mathbf{f}_2(t).\mathbf{v}\, da \quad \forall \mathbf{v} \in V, \tag{36}$$

and note that

$$\mathbf{f} \in L^2(0,T;V'). \tag{37}$$

The adhesion functional $j : L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ defined by

$$j(\beta, \mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} [(p_\nu(u_\nu) - \gamma_\nu \beta^2 R_\nu(u_\nu))v_\nu \ + p_\tau(\beta)\mathbf{R}_\tau(\mathbf{u}_\tau).\mathbf{v}_\tau] da. \tag{38}$$

Keeping in mind (25), (26) and (29) the integrals (38) are well defined. Using standard arguments based on Green's formula (3) to obtain the following variational formulation of the problem (8)-(19) as follows.

*Problem $PV$. Find a displacement field $\mathbf{u} : [0,T] \to V$, a stress field $\sigma : [0,T] \to \mathcal{H}$, a bonding field $\beta : [0,T] \to L^\infty(\Gamma_3)$ and a temperature field $\theta : [0,T] \to E$ such that*

$$\sigma = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \mathcal{F}(\varepsilon(\mathbf{u})) + \int_0^t \mathcal{G}(\sigma(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)), \varepsilon(\mathbf{u}(s)), \theta(s))\, ds \quad \text{in } \Omega \times (0,T)\,, \tag{39}$$

$$(\ddot{\mathbf{u}}(t), \varepsilon(\mathbf{v}))_{V' \times V} + (\sigma(t), \varepsilon(\mathbf{v}))_{\mathcal{H}} + j(\beta(t), \mathbf{u}(t), \mathbf{v}) = (\mathbf{f}(t), \mathbf{v})_{V' \times V}\ \forall \mathbf{v} \in V, \forall t \in [0,T], \tag{40}$$

$$\dot{\theta}(t) + K\,\theta(t) = R\dot{\mathbf{u}}(t) + Q(t) \text{ in } \quad E', \tag{41}$$

$$\dot{\beta}(t) = -(\beta(t)(\gamma_\nu(R_\nu(u_\nu(t)))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_\tau(t)) \mid^2) - \varepsilon_a)_+ \text{ a.e. } t \in (0,T)\,, \tag{42}$$

$$\mathbf{u}(0) = \mathbf{u}_0,\ \dot{\mathbf{u}}(0) = \mathbf{v}_0,\ \beta(0) = \beta_0,\ \theta(0) = \theta_0, \tag{43}$$

where $Q : [0,T] \to E'$, $K : E \to E'$ and $R : V \to E'$ are given by

$$(Q(t), \eta)_{E' \times E} = \int_{\Gamma_3} k_e \theta_R(t) \eta da + \int_\Omega q(t) \eta dx, \tag{44}$$

$$(K\tau, \eta)_{E' \times E} = \sum_{i,j=1}^d \int_\Omega k_{ij} \frac{\partial \tau}{\partial x_j} \frac{\partial \eta}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau.\eta da, \tag{45}$$

$$(R\mathbf{v}, \eta)_{E' \times E} = \int_\Omega r(\mathbf{v}) \eta dx + \int_{\Gamma_3} h_\tau(\mid \mathbf{v}_\tau \mid).\eta da, \tag{46}$$

for all $\mathbf{v} \in V, \eta, \tau \in E$.

Note that the variational problem $PV$ is formulated in terms of displacement field, stress field, temperature field and bonding field. The existence of the unique solution of problem $PV$ is stated and proved in the next section. To this end, the following remark which is used in different places of the paper is given in what follows.

*Remark* 3.1. Note that in the problem $P$ and in the problem $PV$, is not needed to impose explicitly the restriction $0 \le \beta \le 1$. Indeed, equations (42) guarantee that $\beta(\mathbf{x}, t) \le \beta_0(\mathbf{x})$ and, therefore, assumption (42) shows that $\beta(\mathbf{x}, t) \le 1$ for $t \ge 0$, a.e. $\mathbf{x} \in \Gamma_3$. On the other hand, if $\beta(\mathbf{x}, t_0) = 0$ at time $t_0$, then it follows from (42) that $\dot{\beta}(\mathbf{x}, t) = 0$ for all $t \ge t_0$ and therefore, $\beta(\mathbf{x}, t) = 0$ for all $t \ge t_0$, a.e. $\mathbf{x} \in \Gamma_3$. Then $0 \le \beta(\mathbf{x}, t) \le 1$ for all $t \in [0,T]$, a.e. $\mathbf{x} \in \Gamma_3$.

# 4. Existence and Uniqueness Result

The main result in this section is the following existence and uniqueness result.

*Theorem* 4.1. Assume that (22)-(35) hold. Then problem $PV$ has a unique solution $(\mathbf{u}, \sigma, \beta, \theta)$ which satisfies

$$\mathbf{u} \in H^1(0,T;V) \cap C^1(0,T;H),\ \ddot{\mathbf{u}} \in L^2(0,T;V'), \tag{47}$$

$$\sigma \in L^2(0,T;\mathcal{H}),\ Div\, \sigma \in L^2(0,T;V'), \tag{48}$$

$$\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap Z. \tag{49}$$

$$\theta \in C(0,T;L^2(\Omega)) \cap L^2(0,T;E),\ \dot{\theta} \in L^2(0,T;E'). \tag{50}$$

A quadruplet $(\mathbf{u}, \sigma, \beta, \theta)$ which satisfies (39)-(43) is called a weak solution to the contact problem $P$. Then under the stated assumptions, problem (8)-(19) has a unique weak solution satisfying (47)-(50). The proof of Theorem 4.1 will be carried out in several steps and is based on arguments of evolution equations with monotone operators and a fixed point argument. To this end assume in the following that (22)-(43) hold. Below, $C$ denotes a generic positive constant which may depend on $\Omega$, $\Gamma_1, \Gamma_2, \Gamma_3, \mathcal{A}, \mathcal{G}, p_\nu, p_\tau, \gamma_\nu, \gamma_\tau, L$ and $T$ but does not depend on $t$ nor of the rest of input data, and whose value may change from place to place. Moreover, for the sake of simplicity, in what follows, the explicit dependence of various functions on $\mathbf{x} \in \Omega \cup \Gamma$ is suppressed. Let $\eta \in L^2(0,T;V')$ be given, the first step concerns the study of the following variational problem.

*Problem $PV_\eta$. Find a displacement field $\mathbf{u}_\eta : [0,T] \to V$ such that*

$$(\ddot{\mathbf{u}}_\eta(t), \mathbf{v})_{V'\times V} + (\mathcal{A}\varepsilon(\dot{\mathbf{u}}_\eta(t)), \varepsilon(\mathbf{v}))_{\mathcal{H}} + (\eta(t), \mathbf{v})_{V'\times V}$$

$$= (\mathbf{f}(t), \mathbf{v})_{V'\times V} \quad \forall \mathbf{v} \in V, \text{ a.e. } t \in (0,T), \tag{51}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0, \dot{\mathbf{u}}_\eta(0) = \mathbf{v}_0. \tag{52}$$

To solve problem $PV_\eta$, it is important to recall now an abstract existence and uniqueness result, for the convenience of the reader. Let $V$ and $H$ denote real Hilbert spaces such that $V$ is dense in $H$ and the inclusion map is continuous, $H$ is identified with its dual and it is identified with a subspace of the dual $V'$ of $V$, i.e., $V \subset H \subset V'$, and then the inclusions above define a Gelfand triple. The notation $|\,.\,|_V, |\,.\,|_{V'}$ and $(.,.)_{V'\times V}$ represent the norms on $V$ and on $V'$ and the duality pairing between them, respectively. The following abstract result may be found in [26, p. 48].

*Theorem* 4.2. Let $V, H$ be as above, and let $A : V \to V'$ be a hemicontinuous and monotone operator which satisfies

$$(A\mathbf{v}, \mathbf{v})_{V'\times V} \geq \omega \mid \mathbf{v} \mid_V^2 + \lambda \quad \forall \mathbf{v} \in V, \tag{53}$$

$$\mid A\mathbf{v} \mid_{V'} \leq C(\mid \mathbf{v} \mid_V + 1) \quad \forall \mathbf{v} \in V, \tag{54}$$

for some constants $\omega > 0$, $C > 0$ and $\lambda \in \mathbb{R}$. Then, given $\mathbf{u}_0 \in H$ and $\mathbf{f} \in L^2(0,T;V')$, there exists a unique function $\mathbf{u}$ which satisfies

$$\mathbf{u} \in L^2(0,T;V') \cap C(0,T;H),\ \dot{\mathbf{u}} \in L^2(0,T;V'),$$

$$\dot{\mathbf{u}}(t) + A\mathbf{u}(t) = \mathbf{f}(t) \text{ a.e. } t \in (0,T),$$

$$\mathbf{u}(0) = \mathbf{u}_0.$$

Applying it to problem $PV_\eta$.

*Lemma* 4.1. There exists a unique solution to problem $PV_\eta$ and it has the regularity expressed in (47).

*Proof* Define the operator $A : V \to V'$ by

$$(A\mathbf{u}, \mathbf{v})_{V'\times V} = (\mathcal{A}\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}} \quad \forall \mathbf{u}, \mathbf{v} \in V. \tag{55}$$

It follows from (55) and (22)(a) that

$$\mid A\mathbf{u} - A\mathbf{v} \mid_{V'} \leq L_{\mathcal{A}} \mid \mathbf{u} - \mathbf{v} \mid_V \quad \forall \mathbf{u}, \mathbf{v} \in V, \tag{56}$$

which shows that $A : V \to V'$ is continuous, and so is hemicontinuous. Using (55) and (22)(b), to find

$$(A\mathbf{u} - \mathbf{A}\mathbf{v}, \mathbf{u} - \mathbf{v})_{V'\times V} \geq m_{\mathcal{A}} \mid \mathbf{u} - \mathbf{v} \mid_V^2 \quad \forall \mathbf{u}, \mathbf{v} \in V, \tag{57}$$

i.e., that $A : V \to V'$ is a monotone operator. Choosing $\mathbf{v} = \mathbf{0}_V$ in (57) to obtain

$$(A\mathbf{u} - \mathbf{A}\mathbf{v}, \mathbf{u} - \mathbf{v})_{V'\times V} \geq m_{\mathcal{A}} \mid \mathbf{u} - \mathbf{v} \mid_V^2 - \mid A\mathbf{0}_V \mid_{V'} \mid \mathbf{u} \mid_V$$

$$\geq \frac{1}{2} m_{\mathcal{A}} \mid \mathbf{u} \mid_V^2 - \frac{1}{2m_{\mathcal{A}}} \mid A\mathbf{0}_V \mid_{V'}^2, \quad \forall \mathbf{u} \in V.$$

Thus, $A$ satisfies condition (53) with $\omega = \frac{m_{\mathcal{A}}}{2}$ and $\lambda = \frac{-|A\mathbf{0}_V|_{V'}^2}{2m_{\mathcal{A}}}$. Next, using (56) to deduce that

$$\mid A\mathbf{u} \mid_{V'} \leq L_{\mathcal{A}} \mid \mathbf{u} \mid_V + \mid A\mathbf{0}_V \mid_{V'} \quad \forall \mathbf{u} \in V.$$

This inequality implies that $A$ satisfies condition (54). Finally, using (37) and (33) to have $\mathbf{f} - \eta \in L^2(0,T;V')$ and $\mathbf{v}_0 \in H$.

It follows now from Theorem 4.2 that there exists a unique function $\mathbf{v}_\eta$ which satisfies

$$\mathbf{v}_\eta \in L^2(0,T;V) \cap C(0,T;H),\ \dot{\mathbf{v}}_\eta \in L^2(0,T;V'), \tag{58}$$

$$\dot{\mathbf{v}}_\eta(t) + A\mathbf{v}_\eta(t) + \eta(t) = \mathbf{f}(t) \quad \text{a.e. } t \in (0,T), \tag{59}$$

$$\mathbf{v}_\eta(0) = \mathbf{v}_0. \tag{60}$$

Let $\mathbf{u}_\eta : [0,T] \to V$ be the function defined by

$$\mathbf{u}_\eta(t) = \int_0^t \mathbf{v}_\eta(s)\, ds + \mathbf{u}_0 \ \forall t \in [0,T]. \tag{61}$$

It follows from (55) and (58)-(61) that $\mathbf{u}_\eta$ is a solution of the variational problem $PV_\eta$ and it satisfies the regularity expressed in (47). This concludes the existence part of Lemma 4.3. The uniqueness of the solution follows from the uniqueness of the solution to problem (59)-(60), guaranteed by Theorem 4.1.

In the second step, using the displacement field $\mathbf{u}_\eta$ obtained in Lemma 4.3 and considering the following initial-value problem.

*Problem $PV_\beta$. Find the adhesion field $\beta_\eta : [0,T] \to L^2(\Gamma_3)$ such that*

$$\dot{\beta}_\eta(t) = -(\beta_\eta(t)(\gamma_\nu(R_\nu(u_{\eta\nu}(t)))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_{\eta\tau}(t)) \mid^2\ ) - \varepsilon_a)_+ \qquad \text{a.e. } t \in (0,T), \tag{62}$$

$$\beta_\eta(0) = \beta_0. \tag{63}$$

Hence the following result.

*Lemma* 4.2. *There exists a unique solution $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap Z$ to problem $PV_\beta$.*

*Proof* For the sake of simplicity the dependence of various functions on $\Gamma_3$, is suppressed and note that the equalities and inequalities below are valid a.e. on $\Gamma_3$. Consider the mapping $F_\eta : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F_\eta(t,\beta) = -(\beta(\gamma_\nu(R_\nu(u_{\eta\nu}(t)))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_{\eta\tau}(t)) \mid^2\ ) - \varepsilon_a)_+, \tag{64}$$

for all $t \in [0,T]$ and $\beta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operator $R_\nu$ and $\mathbf{R}_\tau$ that $F_\eta$ is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\beta \in L^2(\Gamma_3)$, the mapping $t \to F_\eta(t,\beta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus using a version of Cauchy-Lipschitz theorem given in Theorem 2.1 to deduce that there exists a unique function $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ solution to the problem $PV_\beta$. Also, the arguments used in Remark 3.1 show that $0 \leq \beta_\eta(t) \leq 1$ for all $t \in [0,T]$, a.e. on $\Gamma_3$. Therefore, from the definition of the set $Z$, we find that $\beta_\eta \in Z$, which concludes the proof of the Lemma 4.4.

In the third step, using the displacement field $\mathbf{u}_\eta$ obtained in Lemma 4.3 and consider the following variational problem.

*Problem $PV_\theta$. Find a temperature field $\theta_\eta : [0,T] \to E$ such that*

$$\dot{\theta}_\eta(t) + K\theta_\eta(t) = R\dot{\mathbf{u}}_\eta(t) + Q(t) \text{ in } \quad E', \text{ a.e. } t \in (0,T), \tag{65}$$

$$\theta_\eta(0) = \theta_0. \tag{66}$$

The study of problem $PV_\theta$ is given in the following result.
*Lemma* 4.3. $PV_\theta$ has an unique solution satisfying

$$\theta_\eta \in C(0,T;L^2(\Omega)) \cap L^2(0,T;E),\ \dot{\theta}_\eta \in L^2(0,T;E'). \tag{67}$$

Moreover, there exists $C > 0$ such that $\forall \eta_i \in L^2(0,T;V')$,

$$\mid \theta_1(t) - \theta_2(t) \mid^2_{L^2(\Omega)} \leq C \int_0^t \mid \eta_1(s) - \eta_2(s) \mid^2_{V'} ds,\ \forall t \in [0,T]\,. \tag{68}$$

Here we wrote denote $\theta_{\eta_i} = \theta_i$, for $i = 1,2$.

*Proof* The result follows from classical first order evolution equation given in [2, 26] and proceed like in the proof of Lemma 4.3, where the Gelfand triple is given by

$$E \subset L^2(\Omega) = (L^2(\Omega))' \subset E'.$$

The operator K is linear and coercive. Using Korn's inequality (see [15]) to have

$$(K\tau,\tau)_{E'\times E} \geq C \mid \tau \mid^2_E\,.$$

Here and below, $C > 0$ denotes a generic constant the value of which may change from lines to lines.
Using (65) to deduce that

$$\begin{aligned}&(\dot{\theta}_1 - \dot{\theta}_2, \theta_1 - \theta_2)_{L^2(\Omega)} + (K(\theta_1 - \theta_2), \theta_1 - \theta_2)_{L^2(\Omega)}\\ &= (R(\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2), \theta_1 - \theta_2)_{L^2(\Omega)} \quad \text{a.e. } t \in (0,T)\,.\end{aligned} \tag{69}$$

Integrating the inequality (69) with respect to time, using the initial conditions $\theta_1(0) = \theta_2(0) = \theta_0$, the fact that K is coercive and the Lipschitz continuity of the operator $R$ to find that

$$\frac{1}{2} \mid \theta_1(t) - \theta_2(t) \mid^2_{L^2(\Omega)} \leq L_r \int_0^t \mid \mathbf{v}_1(s) - \mathbf{v}_2(s) \mid_V \mid \theta_1(s) - \theta_2(s) \mid_{L^2(\Omega)} ds.$$

Using the inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, to obtain that

$$\mid \theta_1(t) - \theta_2(t) \mid^2_{L^2(\Omega)} \leq C( \int_0^t \mid \mathbf{v}_1(s) - \mathbf{v}_2(s) \mid_V ds + \int_0^t \mid \theta_1(s) - \theta_2(s) \mid^2_{L^2(\Omega)} ds).$$

Applying Gronwall's inequality [26, p.49] to deduce that

$$\mid \theta_1(t) - \theta_2(t) \mid^2_{L^2(\Omega)} \leq C \int_0^t \mid \mathbf{v}_1(s) - \mathbf{v}_2(s) \mid^2_V ds\ \forall t \in [0,T]\,. \tag{70}$$

Using (51), to find for a.e. $t \in (0,T)$

$$(\dot{\mathbf{v}}_1 - \dot{\mathbf{v}}_2, \mathbf{v}_1 - \mathbf{v}_2)_{V'\times V} + (\mathcal{A}\varepsilon(\mathbf{v}_1) - \mathcal{A}\varepsilon(\mathbf{v}_2), \varepsilon(\mathbf{v}_1 - \mathbf{v}_2))_{\mathcal{H}}. = -(\eta_1 - \eta_2, \mathbf{v}_1 - \mathbf{v}_2)_{V'\times V}.$$

Integrating this equality with respect to time, using the initial conditions $\mathbf{v}_1(0) = \mathbf{v}_2(0) = \mathbf{v}_0$ and condition (22) to find

$$m_{\mathcal{A}} \int_0^t \mid \mathbf{v}_1(s) - \mathbf{v}_2(s) \mid_V^2 \, ds \leq -\int_0^t (\eta_1(s) - \eta_2(s), \mathbf{v}_1(s) - \mathbf{v}_2(s))_{V' \times V} \, ds,$$

for all $t \in [0, T]$. Then, using the inequality $2ab \leq \frac{a^2}{\gamma} + \gamma b^2$ to have

$$\int_0^t \mid \mathbf{v}_1(s) - \mathbf{v}_2(s) \mid_V^2 \, ds \leq C \int_0^t \mid \eta_1(s) - \eta_2(s) \mid_{V'}^2 \, ds \quad \forall t \in [0, T]. \tag{71}$$

The inequality (70) and the relation (71) lead to the estimate (68).

In the fourth step, using the displacement field $\mathbf{u}_\eta$ obtained in Lemma 4.3 and the temperature field $\theta_\eta$ obtained in Lemma 4.5 to construct the following Cauchy problem for the stress field.

*Problem $PV\sigma_\eta$. Find a stress field $\sigma_\eta : [0, T] \to \mathcal{H}$ such that*

$$\sigma_\eta(t) = \mathcal{F}\varepsilon(\mathbf{u}_\eta(t)) + \int_0^t \mathcal{G}(\sigma_\eta(s), \varepsilon(\mathbf{u}_\eta(s)), \theta_\eta(s)) ds \quad \forall t \in [0, T]. \tag{72}$$

The study of problem $PV\sigma_\eta$ is given in the following result.

*Lemma* 4.4. There exists a unique solution of problem $PV\sigma_\eta$ and it satisfies $\sigma_\eta \in W^{1,2}(0, T, \mathcal{H})$. Moreover, if $\sigma_i$, $\mathbf{u}_i$ and $\theta_i$ represent the solutions of problem $PV\sigma_{\eta_i}$, $PV_{\eta_i}$ and $PV_{\theta_i}$, respectively, for $\eta_i \in L^2(0, T; V')$, $i = 1, 2$, then there exists $C > 0$ such that

$$\mid \sigma_1(t) - \sigma_2(t) \mid_{\mathcal{H}}^2 \leq C(\mid \mathbf{u}_1(t) - \mathbf{u}_2(t) \mid_V^2$$

$$+ \int_0^t \mid \mathbf{u}_1(s) - \mathbf{u}_2(s) \mid_V^2 \, ds + \int_0^t \mid \theta_1(s) - \theta_2(s) \mid_{L^2(\Omega)}^2 \, ds) \forall t \in [0, T]. \tag{73}$$

*Proof* Let $\Lambda_\eta : L^2(0, T, \mathcal{H}) \to L^2(0, T, \mathcal{H})$ be the operator given by

$$\Lambda_\eta \sigma(t) = \mathcal{F}\varepsilon(\mathbf{u}_\eta(t)) + \int_0^t \mathcal{G}(\sigma(s), \varepsilon(\mathbf{u}_\eta(s)), \theta_\eta(s)) ds, \tag{74}$$

for all $\sigma \in L^2(0, T, \mathcal{H})$ and $t \in [0, T]$. For $\sigma_1, \sigma_2 \in L^2(0, T, \mathcal{H})$, using (74) and (24) to obtain for all $t \in [0, T]$

$$\mid \Lambda_\eta \sigma_1(t) - \Lambda_\eta \sigma_2(t) \mid_{\mathcal{H}} \leq L_{\mathcal{G}} \int_0^t \mid \sigma_1(s) - \sigma_2(s) \mid_{\mathcal{H}} \, ds.$$

It follows from this inequality that for $p$ large enough, the operator $\Lambda_\eta^p$ is a contraction on the Banach space $L^2(0, T; \mathcal{H})$ and, therefore, there exists a unique element $\sigma_\eta \in L^2(0, T; \mathcal{H})$ such that $\Lambda_\eta \sigma_\eta = \sigma_\eta$. Moreover, $\sigma_\eta$ is the unique solution of problem $PV\sigma_\eta$ and, using (72), the regularity of $\mathbf{u}_\eta$, the regularity of $\theta_\eta$ and the properties of the operators $\mathcal{F}$ and $\mathcal{G}$, it follows that $\sigma_\eta \in W^{1,2}(0, T, \mathcal{H})$. Consider now $\eta_1, \eta_2 \in L^2(0, T; V')$ and for $i = 1, 2$, denote $\mathbf{u}_{\eta_i} = \mathbf{u}_i, \sigma_{\eta_i} = \sigma_i$ and $\theta_{\eta_i} = \theta_i$. Then

$$\sigma_i(t) = \mathcal{F}\varepsilon(\mathbf{u}_i(t)) + \int_0^t \mathcal{G}(\sigma_i(s), \varepsilon(\mathbf{u}_i(s)), \theta_i(s)) \, ds \quad \forall t \in [0, T],$$

and, using the properties (23),(24) on $\mathcal{F}$ and $\mathcal{G}$ to find

$$\mid \sigma_1(t) - \sigma_2(t) \mid_{\mathcal{H}}^2 \leq C(\mid \mathbf{u}_1(t) - \mathbf{u}_2(t) \mid_V^2 + \int_0^t \mid \sigma_1(s) - \sigma_2(s) \mid_{\mathcal{H}}^2 \, ds$$
$$+ \int_0^t \mid \mathbf{u}_1(s) - \mathbf{u}_2(s) \mid_V^2 \, ds + \int_0^t \mid \theta_1(s) - \theta_2(s) \mid_{L^2(\Omega)}^2 \, ds) \quad \forall t \in [0, T].$$

Using Gronwall's argument in the obtained inequality to deduce (73), which concludes the proof of Lemma 4.6.

Finally as a consequence of these results, using the properties of the operator $\mathcal{G}$, the operator $\mathcal{F}$ and the functional $j$, for $t \in [0,T]$, let the element defined by the following equation.

$$(\Lambda\eta(t), \mathbf{v})_{V' \times V} = (\mathcal{F}\varepsilon(\mathbf{u}_\eta(t)), \varepsilon(\mathbf{v}))_{\mathcal{H}} + (\int_0^t \mathcal{G}(\sigma_\eta(s), \varepsilon(\mathbf{u}_\eta(s)), \theta_\eta(s))ds, \varepsilon(\mathbf{v}))_{\mathcal{H}}$$
$$+ j(\beta_\eta(t), \mathbf{u}_\eta(t), \mathbf{v}) \quad \forall \mathbf{v} \in V. \tag{75}$$

Here, for every $\eta \in L^2(0,T;V')$, $\mathbf{u}_\eta, \beta_\eta, \theta_\eta$ and $\sigma_\eta$ represent the displacement field, the bonding field, the temperature field and the stress field obtained in Lemmas 4.3, 4.4, 4.5 and 4.6 respectively. Then the following result is given in what follows.

*Lemma* 4.5. The operator $\Lambda$ has a unique fixed point $\eta^* \in L^2(0,T;V')$ *such that* $\Lambda\eta^* = \eta^*$.

*Proof* Let $\eta_1, \eta_2 \in L^2(0,T;V')$. Using the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i, \dot{\mathbf{u}}_{\eta_i} = \mathbf{v}_{\eta_i} = \mathbf{v}_i, \theta_{\eta_i} = \theta_i$, $\beta_{\eta_i} = \beta_i$ and $\sigma_{\eta_i} = \sigma_i$ for $i = 1,2$. Using (24), (25), (26), the definition of $R_\nu$, $\mathbf{R}_\tau$ and the remark 3.1, to have

$$| \Lambda\eta_1(t) - \Lambda\eta_2(t) |_{V'}^2 \leq C(| \mathbf{u}_1(t) - \mathbf{u}_2(t) |_V^2 + \int_0^t | \sigma_1(s) - \sigma_2(s) |_{\mathcal{H}}^2 \, ds$$
$$+ \int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_V^2 \, ds + \int_0^t | \theta_1(s) - \theta_2(s) |_{L^2(\Omega)}^2 \, ds + \int_0^t | \beta_1(s) - \beta_2(s) |_{L^2(\Gamma_3)}^2 \, ds). \tag{76}$$

Using the estimate (73) to deduce

$$| \Lambda\eta_1(t) - \Lambda\eta_2(t) |_{V'} \leq C(| \mathbf{u}_1(t) - \mathbf{u}_2(t) |_V^2 + \int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_V^2 \, ds$$
$$+ \int_0^t | \theta_1(s) - \theta_2(s) |_{L^2(\Omega)}^2 \, ds + \int_0^t | \beta_1(s) - \beta_2(s) |_{L^2(\Gamma_3)}^2 \, ds).$$

Since

$$\mathbf{u}_i(t) = \int_0^t \mathbf{v}_i(s)ds + \mathbf{u}_0, \quad t \in [0,T],$$

then

$$| \mathbf{u}_1(t) - \mathbf{u}_2(t) |_V^2 \leq C \int_0^t | \mathbf{v}_1(s) - \mathbf{v}_2(s) |_V^2 \, ds \quad \forall t \in [0,T]. \tag{77}$$

On the other hand, using the Cauchy problem (62)-(63) to find

$$\beta_i(t) = \beta_0 - \int_0^t (\beta_i(s)(\gamma_\nu(R_\nu(u_{i\nu}(s)))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_{i\tau}(s)) \mid^2) - \varepsilon_a)_+ \, ds,$$

and then

$$| \beta_1(t) - \beta_2(t) |_{L^2(\Gamma_3)} \leq C \int_0^t | \beta_1(s)(R_\nu(u_{1\nu}(s)))^2 - \beta_2(s)(R_\nu(u_{2\nu}(s)))^2 |_{L^2(\Gamma_3)} \, ds$$
$$+ C \int_0^t | \beta_1(s) \mid \mathbf{R}_\tau(\mathbf{u}_{1\tau}(s)) \mid^2 - \beta_2(s) \mid \mathbf{R}_\tau(\mathbf{u}_{2\tau}(s)) \mid^2 |_{L^2(\Gamma_3)} \, ds.$$

Using the definition of $R_\nu$ and $\mathbf{R}_\tau$ and writing $\beta_1 = \beta_1 - \beta_2 + \beta_2$, to get

$$| \beta_1(t) - \beta_2(t) |_{L^2(\Gamma_3)} \leq C(\int_0^t | \beta_1(s) - \beta_2(s) |_{L^2(\Gamma_3)} \, ds + \int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_{L^2(\Gamma_3)^d} \, ds). \tag{78}$$

Next, applying Gronwall's inequality to deduce

$$| \beta_1(t) - \beta_2(t) |_{L^2(\Gamma_3)} \leq C \int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_{L^2(\Gamma_3)^d} \, ds,$$

and the relation (21) leads to

$$| \beta_1(t) - \beta_2(t) |_{L^2(\Gamma_3)}^2 \leq C \int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_V^2 \, ds. \tag{79}$$

Substituting (79) in (76) and using (77) to obtain

$$| \Lambda\eta_1(t) - \Lambda\eta_2(t) |_{V'}^2 \leq C(| \mathbf{u}_1(t) - \mathbf{u}_2(t) |_V^2 + \int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_V^2 \, ds + \int_0^t | \theta_1(s) - \theta_2(s) |_{L^2(\Omega)}^2 \, ds)$$

$$\leq C(\int_0^t | \mathbf{v}_1(s) - \mathbf{v}_2(s) |_V^2 \, ds + \int_0^t | \theta_1(s) - \theta_2(s) |_{L^2(\Omega)}^2 \, ds).$$

It follows now from the previous inequality, the estimates (71) and (68) that

$$| \Lambda\eta_1(t) - \Lambda\eta_2(t) |_{V'}^2 \leq C \int_0^t | \eta_1(s) - \eta_2(s) |_{V'}^2 \, ds.$$

Reiterating this inequality $m$ times leads to

$$| \Lambda^m\eta_1 - \Lambda^m\eta_2 |_{L^2(0,T;V')}^2 \leq \frac{C^m T^m}{m!} | \eta_1 - \eta_2 |_{L^2(0,T;V')}^2 .$$

Thus, for m sufficiently large, $\Lambda^m$ is a contraction on the Banach space $L^2(0,T;V^{'})$, and so $\Lambda$ has a unique fixed point.

Now, all the ingredients needed to prove Theorem 4.1 are satisfied.

*Proof* Let $\eta^* \in L^2(0,T;V^{'})$ be the fixed point of $\Lambda$ given by (75). Denote by $\mathbf{u} = \mathbf{u}_{\eta^*}$ the solution of the problem $PV_\eta$ for $\eta = \eta^*$, $\theta = \theta_{\eta^*}$ the solution of the problem $PV_\theta$ for $\eta = \eta^*$and $\beta = \beta_{\eta^*}$ the solution of the problem $PV_\beta$ for $\eta = \eta^*$. Let $\sigma_{\eta^*}$ be the solution of the problem $PV_\sigma$ for $\eta = \eta^*$, denote

$$\sigma = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \sigma_{\eta^*}.$$

Using (75) and keeping in mind that $\Lambda\eta^* = \eta^*$, to find that the quadruplet $(\mathbf{u}, \sigma, \beta, \theta)$ is a solution of the problem $PV$. This solution has the regularity expressed in (47)-(50) and which follow from the regularities of the solution of problems $PV_\eta$, $PV_\beta, PV_\theta$ and $PV_\sigma$. Moreover, it follows from (47), (22) and (24) that $\sigma \in L^2(0,T;\mathcal{H})$. Choosing now $\mathbf{v} = \pm\varphi$ in (40), where $\varphi \in C_0^\infty(\Omega)^d$, and using (28), (38) to find

$$\rho\ddot{\mathbf{u}}(t) = Div \; \sigma(t) + \mathbf{f}_0(t) \quad \text{a.e. } t \in (0,T) \, .$$

Now assumptions (28), (30), the fact that $\ddot{\mathbf{u}} \in L^2(0,T;V^{'})$ and the above equality imply that $Div \; \sigma \in L^2(0,T;V^{'})$, which shows that $\sigma$ satisfies (48).

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator $\Lambda$ defined by (4.29) and the unique solvability of problem $PV_\eta, PV_\beta, PV_\theta$ and $PV_{\sigma_\eta}$ .

# 5. Conclusion

This paper is devoted to the study of a frictionless contact problem coupling two importants phenomena which are temperature of the material and the adhesion of the contact surface without adhesive wear. The evolution of the adhesive wear is a big challenge which may be explored in future.

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